Markov Processes – Problem Sheet 4.

Tutorials by Nikolay Barashkov <s6nibara@uni-bonn.de>, Robert Crowell <crowellr@googlemail.com>.
Solutions will be collected Tuesday November 21st during the lecture. At most in groups of 2.

Exercise 1. [5pt] Consider the Markov chain on $\mathbb{R}^d$ given by the recurrence

$$X_{n+1} = X_n + b(X_n) + \sigma(X_n) W_{n+1},$$

where $b: \mathbb{R}^d \to \mathbb{R}^d$ and $\sigma: \mathbb{R}^d \to \mathbb{R}^{d \times d}$ are measurable functions and (W_n) is an iid sequence of square integrable random variable such that $\mathbb{E}(W) = 0$ and $\mathrm{Cov}(W_i, W_j) = \delta_{ij}$. Show by considering a Lyapunov function $V(x) = |x|^2/\varepsilon$ with $\varepsilon > 0$, that sufficiently large balls are positive recurrent provided

$$\limsup_{|x| \to +\infty} (2x \cdot b(x) + |b(x)|^2 + \mathrm{tr}(\sigma^T(x)\sigma(x))) < 0.$$

Exercise 2. (Recurrence of Brownian motion) [5pt] Brownian motion $(B_t)_t$ on $\mathbb{R}^d$ solves the (continuous time) martingale problem for the generator $\mathcal{L} = \frac{1}{2}\Delta$ with domain $f \in C_b^2(\mathbb{R}^d)$ (twice differentiable, bounded functions), it the sense that for such f the process

$$M_t^f = f(B_t) - \frac{1}{2}\int_0^t \Delta f(X_s) \mathrm{d}s$$

is a martingale wrt $(\mathcal{F}_t^B)_t$. Let $T_a = \inf\{t \geqslant 0: |B_t| = a\}$.

a) Compute $\mathbb{P}_x(T_a < T_b)$ for $a < |x| < b$;

b) Show that when $d \leqslant 2$, $\mathbb{P}_x(T_a < \infty) = 1$ for any $a < |x|$;

c) Show that when $d \geqslant 3$, $\mathbb{P}_x(T_a < 1) \to 0$ as $|x| \to 0$.

(You may assume the optional stopping theorem and the martingale convergence theorem in continuous time without proof)

Exercise 3. (Tightness) [5pt] Prove the following three statements.

a) A sequence of probability measures on the line is tight if and only if, for the corresponding distribution functions F_n, we have $\lim_{x \to \infty} F_n(x) = 1$ and $\lim_{x \to -\infty} F_n(x) = 0$ uniformly in n.

b) A sequence of normal distributions on the line is tight if and only if the means and the variances are bounded (a normal distribution with variance 0 being a point mass).

c) A sequence of distributions of random variables X_n is tight if (X_n) is uniformly integrable. Recall: A sequence of random variables X_n is uniformly integrable if, when $L \to \infty$

$$\sup_n \mathbb{E}[|X_n|; |X_n| \geq L] \to 0.$$

Exercise 4. (CYCLES) [8pt] Let E be a countable state space and consider a Markov chain $(X_n)_n$ on E with transition kernel π. Let

$$N_x^y = \sum_{n=0}^{T_x^+ - 1} \mathbb{I}_{X_n = y}$$

the r.v. which counts the visits to y before a return to x. Recall that $T_x^+ = \inf\{n > 1: X_n = x\}$. Let $\mu_x(y) = \mathbb{E}_x[N_x^y]$ seen as a positive measure on E via $\mu_x(A) = \sum_{y \in A} \mu_x(y)$ for any $A \subseteq E$.

a) Show that x is positive recurrent iff μ_x has finite mass.

b) Show that if $x \in E$ is a recurrent state then μ_x is an invariant measure for π (not necessarily a probability).

c) Show that if the chain is irreducible and recurrent then $\mu_x(y) < \infty$ for all $x, y \in E$ (hint: use the invariance of μ_x and the fact that $\pi^k(x, y) > 0$ for some k). Deduce that if E is finite, then the chain is positive recurrent.

d) Assume the chain is irreducible and positive recurrent. Assume also that there exists only one invariant probability ν. Deduce that $\nu(x) = 1 / \mathbb{E}_x[T_x]$ for all $x \in E$ and that

$$\mathbb{E}_x[N_x^y] = \nu(y) / \nu(x).$$