Markov Processes – Problem Sheet 6.

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Solutions will be collected Tuesday December 5th during the lecture. At most in groups of 2.

Exercise 1. (Ergodicity) [2 pts] Let $((X_n)_{n\in\mathbb{N}}, \mathbb{P})$ be a canonical stationary process with state space E, and let $\mathcal{J} = \{A\colon A = \Theta^{-1}(A)\}$ be the σ-algebra of shift invariant events. Prove that for $F\colon \Omega \to \mathbb{R}$ the following two properties are equivalent:

i. F is $\mathcal{J}$-measurable.

ii. $F = F \circ \Theta$

Conclude that $\mathbb{P}$ is ergodic if and only if any shift-invariant function $F\colon \Omega \to \mathbb{R}$ is $\mathbb{P}$-almost surely constant.

Exercise 2. (Rotations of the circle) [3 pts] Let $\Omega = \mathbb{R}/\mathbb{Z} = [0,1]/\sim$ where $0 \sim 1$. Consider the rotation $\theta(\omega) = \omega + a \pmod 1$ with $a = p/q$, $p, q \in \mathbb{N}$ relatively primes.

a) Show that for any $x \in \Omega$, the uniform distribution $\mathbb{P}_x$ on $\{x, x+a, x+2a, \ldots, x+(q-1)a\}$ is θ-invariant and ergodic.

b) Determine all θ-invariant probability measures on Ω, and represent them as a mixture of ergodic ones.

Exercise 3. (Ergodicity for Markov chains) [5 pts] Suppose that $((X_n)_n, \mathbb{P}_x)$ is a canonical time–homogeneous Markov chain, and let μ be an invariant probability measure for the transition kernel π. Show that the following properties are all equivalent:

i. $\mathbb{P}_\mu$ is ergodic.

ii. $\frac{1}{n}\sum_{i=1}^{n-1} F \circ \Theta^i \to \mathbb{E}_\mu[F]$, $\mathbb{P}_\mu$-almost surely for any $F\colon \Omega \to \mathbb{R}$ such that $\mathbb{E}_\mu|F| < \infty$.

iii. $\frac{1}{n}\sum_{i=1}^{n-1} f(X_i) \to \mu(f)$, $\mathbb{P}_\mu$-almost surely for any $f\colon E \to \mathbb{R}$ such that $\mu(|f|) < \infty$.

iv. For any $B \in \mathcal{E}$, $\frac{1}{n}\sum_{i=1}^{n-1} \pi^n(x, B) \to \mu(B)$, for μ-a.e. $x \in \mathcal{E}$.

v. For any $B \in \mathcal{E}$ such that $\mu(B) > 0$, $\mathbb{P}_x(T_B < \infty) > 0$ for μ–a.e. $x \in \mathcal{E}$.

vi. Any set $B \in \mathcal{E}$ satisfying $\pi \mathbb{I}_B = \mathbb{I}_B$ μ-almost surely has measure $\mu(B) \in \{0, 1\}$.

Conclude that (i)-(vi) are satisfied if π is μ-irreducible.

Exercise 4. (Ergodicity and irreducibility for Markov processes) [5 pts] Consider a canonical Markov process $((X_t)_{t\geqslant 0}, \mathbb{P}_x)$ with state space $(E, \mathcal{E})$ and transition semigroup $(p_t)_{t\geqslant 0}$.

a) Show that for $\mu \in \mathcal{P}(E)$, the following three conditions are equivalent:

   i. $\mathbb{P}_\mu \circ \theta_t^{-1} = \mathbb{P}_\mu$ for any $t \geqslant 0$;

   ii. $((X_t)_{t\geqslant 0}, \mathbb{P}_x)$ is a stationary process;

   iii. μ is invariant with respect to p_t for any $t \geqslant 0$.

b) Show that the following three conditions are equivalent:

   i. $\mathbb{P}_\mu$ is ergodic;

   ii. Every function $h \in \mathcal{L}^2(\mu)$ such that $p_t h = h$ μ-a.s. $\forall t \geqslant 0$ is almost surely constant.

   iii. Every set $B \in E$ such that $p_t \mathbb{I}_B = \mathbb{I}_B$ μ-a.s. for any $t \geqslant 0$ satisfies $\mu(B) \in \{0, 1\}$.

c) Show that for any shift-invariant event A, there exists $B \in E$ with $p_t \mathbb{I}_B = \mathbb{I}_B$ μ-a.s. for any $t \geqslant 0$ such that $\mathbb{I}_A = \mathbb{I}_{\{X_0 \in B\}}$ $\mathbb{P}_\mu$-a.s.

Exercise 5. (Ergodicity and decay of correlations). [5 pts] We consider a stationary stochastic process $(X_t)_t \in [0, \infty)$ defined on the canonical probability space $(\Omega, \mathcal{F}, \mathbb{P})$.

a) Prove that the following properties are equivalent:

   i. $\mathbb{P}$ is ergodic;

   ii. $\mathrm{Var}\left[\frac{1}{t}\int_0^t F \circ \theta_s \mathrm{d}s\right] \to 0$ as $t \uparrow \infty$ for all $F \in \mathcal{L}^2(\mathbb{P})$;

   iii. $\mathrm{Cov}\left[\frac{1}{t}\int_0^t F \circ \theta_s \mathrm{d}s, G\right] \to 0$ as $t \uparrow \infty$ for all $F, G \in \mathcal{L}^2(\mathbb{P})$;

   iv. $\mathrm{Cov}\left[\frac{1}{t}\int_0^t F \circ \theta_s \mathrm{d}s, F\right] \to 0$ as $t \uparrow \infty$ for all $F \in \mathcal{L}^2(\mathbb{P})$;

b) The process (X_t) is said to be *mixing* iff $\lim_{t\to\infty} \mathrm{Cov}(F \circ \theta_t, G) = 0$ for all $F, G \in \mathcal{L}^2(\mathbb{P})$. Prove that:

   i. If $(X_t)_{t\geqslant 0}$ is mixing then it is ergodic.

   ii. If the tail field $\mathcal{A} = \cap_{t\geqslant 0} \sigma(X_s : s \geqslant t)$ is trivial then $(X_t)_{t\geqslant 0}$ is mixing (and hence ergodic).