Institute for Applied Mathematics – **WS17/18**
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Markov Processes – Problem Sheet 7.

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Solutions will be collected Tuesday December 12th during the lecture. At most in groups of 2.

Exercise 1. (Ergodic transformations) [5pts] Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and θ a transformation of Ω which preserves $\mathbb{P}$.

a) Consider the circle $\Omega = \mathbb{T} = \mathbb{R}/\mathbb{Z}$ with Lebesgue measure $\mathbb{P}$. Prove that the doubling transformation $\theta(x) = 2x \bmod 1$ is ergodic.

b) Given an example showing that the tensor product of two ergodic transformations is not necessarily ergodic (in the product space, with the product measure).

c) Consider the two dimensional torus $\Omega = \mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$ with Lebesgue measure $\mathbb{P}$ and for $\alpha \in \mathbb{R}^2$ the transformation $\theta: \Omega \to \Omega$ given by $\theta(x) = (x_1 + \alpha_1 \bmod 1, x_2 + \alpha_2 \bmod 2)$. Find an arithmetic condition on α which guarantees ergodicity of θ.

d) Prove that the operator $Uf = f \circ \theta$ on $L^2(\mathbb{P})$ is unitary and ergodicity is equivalent to the fact that U has a unique eigenfunction with eigenvalue 1.

e) Prove that θ is ergodic if and only if any measurable function $f: \Omega \to \mathbb{R}$ with $f(\theta(\omega)) \geqslant f(\omega)$ almost everywhere is equal to a constant almost everywhere.

Exercise 2. (Poincaré recurrence) [5pts] Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and θ a transformation of Ω which preserves $\mathbb{P}$ and let $B \in \mathcal{F}$.

a) Prove that there exists a measurable set $F \subseteq E$ such that $\mathbb{P}(E) = \mathbb{P}(F)$ and for which $\omega \in F \Rightarrow \theta^n(\omega) \in E$ infinitely often.

(Hint: consider the set $B = \{\omega \in E: \theta^n(\omega) \notin E \text{ for all } n \geqslant 1\}$. Show that the sets $B_k = \theta^{-k}(B)$ are disjoint and conclude that $\mathbb{P}(B) = 0$. From this deduce that we have to come back to E at least once. Repeat the argument for iterates of θ and conclude.)

b) Show that $\mathbb{P}$ being a finite measure is a necessary condition for the existence of such a set F.

Exercise 3. (Brownian motion on $\mathbb{T}$) [5pts] A Brownian motion $(X_t)_t$ on the circle $\mathbb{T} = \mathbb{R}/\mathbb{Z}$ can be obtained by considering a Brownian motion $(B_t)_t$ on $\mathbb{R}$ modulo the integers, i.e.

$$X_t = B_t - \lfloor B_t \rfloor \in [0, 1) \simeq \mathbb{R}/\mathbb{Z}.$$

Prove the following statements:

a) The process $(X_t)_t$ is a Markov process with transition density w.r.t. the uniform distribution given by

$$p_t(x, y) = \frac{1}{\sqrt{2\pi t}} e^{-\frac{|x-y-n|^2}{2t}}, \qquad \text{for any } t > 0 \text{ and } x, y \in [0, 1).$$

b) For any initial condition, $(X_t)_t$ solves the martingale problem for the operator $\mathcal{L}f = f''/2$ defined on $C^\infty(\mathbb{R}/\mathbb{Z})$. (Observe that there is a one to one correspondence between smooth functions on $\mathbb{R}/\mathbb{Z}$ and periodic smooth funcitons on $\mathbb{R}$)

c) The uniform distribution μ is an invariant probability measure for $(p_t)_t$ and the process with initial distribution μ is stationary and ergodic.

d) The generator $\mathcal{L}$ has smooth, real valued eigenfunctions $(e_n)_{n\in\mathbb{Z}}$, with corresponding eigenvalues $\lambda_n = 2\pi^2 n^2$. Moreover $p_t e_n = \exp(-\lambda_n t) e_n$ for any $t \geqslant 0$.

e) For any $f \in L^2(\mu)$

$$\|p_t f - \mu(f)\|_{L^2(\mu)} \leqslant e^{-2\pi^2 t} \mathrm{Var}_\mu(f).$$

f) Conclude that, for the process with initial distribution μ,

$$\mathbb{E}\left[\left(\frac{1}{t}\int_0^t f(X_s)\mathrm{d}s - \mu(f)\right)^2\right] \leqslant \frac{1}{\pi^2 t}\mathrm{Var}_\mu(f)$$

for any $t \geqslant 0$ and $f \in L^2(\mu)$.

Exercise 4. (Metropolis–Hastings method) [5pts] Let $\mu(\mathrm{d}x) = \mu(x)\mathrm{d}x$ be a probability measure on $\mathbb{R}^d$ with strictly positive density, and let $q(x, \mathrm{d}y) = q(x, y)\mathrm{d}y$ be a probability kernel on $\mathbb{R}^d$ with strictly positive density. The Metropolis-Hastings acceptance probability is given by

$$\alpha(x, y) = \min\left(1, \frac{\mu(y)q(y, x)}{\mu(x)q(x, y)}\right), \qquad x, y \in \mathbb{R}^d.$$

Metropolis–Hastings algorithm

For $n = 0$ choose a point $X_0 \in \mathbb{R}^d$. For $n > 0$ sample Y_n according to $q(X_{n-1}, \cdot)$ and U_n uniformly in $[0, 1]$ independently and let $X_n = Y_n$ if $U_n < \alpha(X_{n-1}, Y_n)$ otherwise let $X_n = X_{n-1}$.

Show that for any bounded measurable function $f: \mathbb{R}^d \to \mathbb{R}$ we have

$$\frac{1}{n}\sum_{k=0}^{n-1} f(X_k) \to \mu(f), \qquad \text{almost surely.}$$