Institute for Applied Mathematics – WS17/18
Massimiliano Gubinelli / Nikolay Barashkov / Robert Crowell

Markov Processes – Problem Sheet 8.

Tutorials by Nikolay Barashkov <s6nibara@uni-bonn.de>, Robert Crowell <crowellr@googlemail.com>.
Solutions will be collected Tuesday December 19th during the lecture. At most in groups of 2.

Exercise 1. (Asymptotic variances of ergodic averages) [5pts] We consider a stationary Markov chain $(X_n, \mathbb{P}_\mu)$ with state space $(E, \mathcal{E})$, transition kernel π, and initial distribution μ.

a) For $f \in L^2(\mu)$ let $f_0 = f - \mu(f)$ and let $A_t f = t^{-1}\sum_{k=0}^{n-1} f(X_k)$. Prove (without assuming the CLT) that if $G f_0 = \sum_{k=0}^{\infty} \pi^k f_0$ converges in $L^2(\mu)$ then

$$\lim_{t\to\infty} t \operatorname{Var}(A_t f) = 2(f_0, G f_0)_{L^2(\mu)} - (f_0, f_0)_{L^2(\mu)} = \operatorname{Var}_\mu(f) + \sum_{k=1}^{\infty} \operatorname{Cov}_\mu(f, \pi^k f).$$

b) Let $E = \{1, 2\}$ and $\pi(1,1) = \pi(2,2) = p$ and $\pi(2,1) = \pi(1,2) = 1 - p$ with $p \in (0,1)$. Show that the unique stationary distribution μ is given by $\mu(1) = \mu(2) = 1/2$ for all values of p. Now consider $S_n = A_n - B_n$ where $A_n = \#\{k \leqslant n: X_k = 1\}$ and $B_n = \#\{k \leqslant n: X_k = 2\}$. Show that $S_n/\sqrt{n}$ satisfies a CLT and compute the limiting variance $\sigma^2(p)$ as a function of p. How does this variance behaves as $p \to 0$? Can you explain it? What is the value of $\sigma^2(1/2)$? Could you have guessed it?

Exercise 2. (Random walks on $\mathbb{Z}_+$) [5pts] Let $\delta \in (0, 1)$. We consider a random walk on the nonnegative integers with transition probabilities

$$\pi(x, y) = \frac{1}{2}(\mathbb{I}_{x=y} + \mathbb{I}_{x=0,y=1}) + \frac{(1-\delta)}{4}\mathbb{I}_{y=x+1,x\geqslant 1} + \frac{(1+\delta)}{4}\mathbb{I}_{y=x-1,x\geqslant 1}$$

a) Find the invariant probability measure μ explicitly.

b) Let $f: \mathbb{Z}_+ \to \mathbb{R}$ be a function with compact support. Solve the Poisson equation $-\mathcal{L}g = f$ explicitly (e.g. using the Ansatz $g = uh$ where h is a solution to $\mathcal{L}h = 0$). Show that for large x a solution g either grows exponentially, or it is a constant.

c) Show that there is a solution g that is a constant for large x if and only if $\mu(f) = 0$. What can you say about the asymptotic variance and the central limit theorem for $\sum_{k=0}^{n-1} f(X_k)$ for such functions f?

Exercise 3. (Couplings on $\mathbb{R}^d$) [5pts] Let $W: \Omega \to \mathbb{R}^d$ be a random variable on $(\Omega, \mathcal{F}, \mathbb{P})$, and let $\mu_a = \operatorname{Law}(a + W)$.

a) (Synchronous coupling) Let $X = a + W$ and $Y = b + W$ for $a, b \in \mathbb{R}^d$. Show that

$$\mathcal{W}^2(\mu_a, \mu_b) = |a - b| = [\mathbb{E}(|X - Y|^2)]^{1/2},$$

i.e. that (X, Y) is an optimal coupling wrt. $\mathcal{W}^2$.

b) (Reflection coupling) Assume that $\mathrm{Law}(W) = \mathrm{Law}(-W)$. Let $\tilde{Y} = \tilde{W} + b$ where $\tilde{W} = W - 2(e \cdot W)e$ with $e = (a - b) / |a - b|$. Prove that $(X, \tilde{Y})$ is a coupling of μ_a and μ_b and if $|W| \leqslant |a - b|/2$ a.s. then

$$\mathbb{E}(f(|X - \tilde{Y}|)) \leqslant f(|a - b|) = \mathbb{E}[f(|X - Y|)]$$

for any concave, increasing function $f: \mathbb{R}_+ \to \mathbb{R}_+$ such that $f(0) = 0$.

Exercise 4. (Structure of invariant measures) [5pts] Let π be a probability kernel on $(E, \mathcal{E})$ and let $\mathcal{S}(\pi) = \{\mu \in \mathcal{P}(E): \mu\pi = \mu\}$.

a) Show that $\mathcal{S}(\pi)$ is convex.

b) Prove that $\mu \in \mathcal{S}(\pi)$ is extremal if and only if every set $B \in \mathcal{E}$ such that $\pi \mathbb{I}_B = \mathbb{I}_B$ μ–a.e. satisfies $\mu(B) \in \{0, 1\}$.

c) Show that every $\mu \in \mathcal{S}(\pi)$ is a convex combination of extremals *(Hint: you can use exercise* 4c *of Sheet 6).*