Institute for Applied Mathematics – **WS17/18**
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Markov Processes – Problem Sheet 9.

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Solutions will be collected Tuesday January 9th during the lecture. At most in groups of 2.

Exercise 1. (Total variation distances) [5pts]

a) Let $\nu = \otimes_{i=1}^{d} \nu_i$ and $\mu = \otimes_{i=1}^{d} \mu_i$ be two product probability measures on E^d. Show in at least two different ways that $\|\nu - \mu\|_{\mathrm{TV}} \leqslant \sum_{i=1}^{d} \|\nu_i - \mu_i\|_{\mathrm{TV}}$.

b) Show that the total variation distance of the law of a Markov chain to its stationary distribution is a non-increasing function of time.

c) Do similar statements as in a) and b) hold when the total variation distance is replaced by a general transportation metric $\mathcal{W}^1$?

Definition 1. *Let $\varepsilon > 0$. The ε-mixing time of a Markov semigroup $(p_t)_t$ on E with invariant probability measure μ is defined as* $t_{\mathrm{mix}}(\varepsilon) := \inf \{t \geq 0 : \|p_t(x, \cdot) - \mu\|_{\mathrm{TV}} \leqslant \varepsilon, \quad \forall x \in E\}$.

Exercise 2. (Hard core model) [5pts] Consider a finite graph (V, E) with n vertices of maximal degree Δ. The corresponding hard core model with fugacity $\lambda > 0$ is the probability measure μ_λ on $\{0, 1\}^V$ with mass function

$$\mu_\lambda(\eta) = \begin{cases} \frac{\lambda^{\sum_{x \in V} \eta(x)}}{Z(\lambda)}, & \text{if } \eta(x)\,\eta(y) = 0 \text{ for any } (x, y) \in E, \\ 0 & \text{otherwise} \end{cases}$$

where $Z(\lambda)$ is a normalization constant.

a) Describe the transition rule for the Glauber dynamics with equilibrium μ_λ, and determine the transition kernel π.

b) Prove that for $\lambda < (\Delta - 1)^{-1}$ and $t \in \mathbb{N}$,

$$\mathcal{W}^1(\nu \pi^t, \mu) \leqslant \alpha(n, \Delta)^t \mathcal{W}^1(\nu, \mu) \leqslant \exp\left(-\frac{t}{n}\left(\frac{1 - \lambda(\Delta - 1)}{1 + \lambda}\right)\right) \mathcal{W}^1(\nu, \mu),$$

where $\alpha(n, \Delta) = 1 - \frac{1}{n}\left(\frac{1 - \lambda(\Delta - 1)}{1 + \lambda}\right)$ and $\mathcal{W}^1$ is the Wasserstein metric based on the Hamming distance on $\{0, 1\}^V$.

c) Show that in this case, the ε-mixing time is of order $O(n \log n)$ for any $\varepsilon \in (0, 1)$.

Exercise 3. (Conductance and lower bounds for mixing times) [5pts] Let π be a transition kernel on $(E, \mathcal{E})$ with stationary distribution μ. For sets $A, B \in \mathcal{E}$ with $\mu(A) > 0$, the equilibrium flow $Q(A, B)$ from A to B is defined by

$$Q(A, B) = (\mu \otimes \pi)(A \times B) = \int_A \mu(\mathrm{d}x) \pi(x, B),$$

and the *conductance* of A is given by $\Phi(A)=\frac{Q(A,A^c)}{\mu(A)}$. The *bottleneck ratio* (isoperimetric constant) Φ_* is defined as $\Phi_* = \min_{A:\mu(A)\leqslant 1/2}\Phi(A)$. Let $\mu_A(B)=\mu(B|A)$ denote the conditioned measure on A.

a) Show that for any $A \in B$ with $\mu(A)>0$,

$$\|\mu_A\pi - \mu_A\|_{TV} = (\mu_A\pi)(A^c) = \Phi(A).$$

[Hint: prove first that

i. $\mu_A\pi(B) - \mu_A(B) \leqslant 0$ *for any measurable* $B \subseteq A$, *and*

ii. $\mu_A\pi(B) - \mu_A(B) = \mu_A\pi(B) \geqslant 0$ *for any measurable* $B \subseteq A^c$.*]*

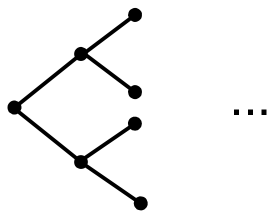
b) Conclude that

$$\|\mu_A - \mu\|_{\mathrm{TV}} \leqslant t\Phi(A) + \|\mu_A\pi^t - \mu\|_{\mathrm{TV}}, \qquad \text{for any } t \in \mathbb{Z}_+.$$

c) Hence prove the lower bound

$$t_{\mathrm{mix}}\left(\frac{1}{4}\right) \geqslant \frac{1}{4\Phi_*}.$$

Exercise 4. (Lazy random walk on a binary tree) [5pts]



Consider the lazy random walk with resting probability $\pi(x,x)=1/2$ on a binary tree of depth k. Let $m=2^{k+1}-1$ denote the number of vertices. Prove that:

a) $\limsup_{m\to\infty} t_{\mathrm{mix}}(1/4)/m < \infty$,

b) $\liminf_{m\to\infty} t_{\mathrm{mix}}(1/4)/m > 0$.

[Hint: You may assume the conductance bound from the previous exercise!]