Sheet 2

(Lectures of 17/4/2018 and 19/4/2018)

1 More on the law of r.v. in a quantum probability space

▷ **characteristic functions?** Let $z = x + iy = re^{i\varphi} \in \mathbb{C}$ and consider the skew–adjoint operator $J = i(x\sigma_x + y\sigma_y)$. A direct computation gives $J^2 = -r^2$ and therefore

$$\exp(J) = \sum_{n\geqslant 0} \frac{J^n}{n!} = \sum_{n\geqslant 0} \frac{J^{2n}}{(2n)!} + \sum_{n\geqslant 0} \frac{J^{2n+1}}{(2n+1)!} = \sum_{n\geqslant 0} \frac{(-1)^n r^{2n}}{(2n)!} + i(\cos\varphi\sigma_x + \sin\varphi\sigma_y) \sum_{n\geqslant 0} \frac{(-1)^n r^{2n+1}}{(2n+1)!}$$

$$= \cos(r) + i(\cos\varphi\sigma_x + \sin\varphi\sigma_y)\sin(r) = \begin{pmatrix} \cos(r) & ie^{-i\varphi}\sin(r) \\ ie^{i\varphi}\sin(r) & \cos(r) \end{pmatrix}.$$

And we can try to define a characteristic function for the pair of random variables σ_x, σ_y by

$$\varphi(x, y) = \mathrm{Tr}[\,\rho \exp[i(x\sigma_x + y\sigma_y)]].$$

Taking for example $\rho = |e_0\rangle\langle e_0|$ with $e_0 = (1, 0)$ we get $\varphi(x, y) = \cos((x^2 + y^2)^{1/2})$. This function cannot be the caracteristic function of any measure on $\mathbb{R}^2$. Indeed $\varphi(t, 0) = \varphi(0, t) = \cos(|t|)$, so both marginals of this measure would have been concentrated on the set $\{-1, 1\}$ which is only possible if the measure itself is concentrated on $\{-1, 1\}^2$. Then it is easy to see that there are no measure on this set which has the above function as characteristic function. This is a very basic example of the fact that non-commuting observables do not have a joint law.

▷ Also observe that while σ_x and σ_y takes values ± 1 the random variable $\sigma_x + \sigma_y$ takes values $\pm\sqrt{2}$! Clearly no classical interpretation of this is possible.

▷ Let $\mathcal{H}$ be an Hilbert space of dimension n and let u a fixed unit vector. Consider an operator A which has spectrum $\{0, \ldots, n-1\}$. In the pure state u is has law $\mu(k) = \langle u|\mathbb{I}_{A=k}|u\rangle$. Now let $(p_k)_k$ an arbitrary law on $\{0, \ldots, n-1\}$ and consider a vector v such that $p(k) = \langle v|\mathbb{I}_{A=k}|v\rangle$ which can always be constructed. We can also construct a unitary operaror U such that $Uu = v$. Let $B = U^{-1}AU$ and observe that $\langle u|\mathbb{I}_{B=k}|u\rangle = \langle u|U^{-1}\mathbb{I}_{A=k}U|u\rangle = \langle Uu|\mathbb{I}_{A=k}|Uu\rangle = \langle v|\mathbb{I}_{A=k}|v\rangle = p(k)$. The random variable B has law p in the pure state u. In particular for any law on $\{0, \ldots, n-1\}$ we can construct a random variable with this given law in the fixed quantum probability space $(\mathcal{H}, u)$. This is clearly not possible in a finite classical probability space.

▷ **A deterministic quantum process.** Deterministic time evolution is represented by a family $(U_t)_t$ of unitary operators on $\mathcal{H}$ such that $U_tU_s = U_{t+s}$. Expectation values at time t are given by $\mathrm{Tr}[\rho U_t^{-1}XU_t]$. Random variables evolve in time as $X_t = U_t^{-1}XU_t$. Take the Bernoulli space, $U_t = \exp(it\sigma_z/2)$ and $X_t = \exp(-it\sigma_z/2)\sigma_x\exp(it\sigma_z/2)$ in the state $\rho = \mathbb{I}/2$. Then the covariance between different times is

$$\mathrm{Tr}[\rho X_t X_s] = \cos(t - s).$$

So $(X_t)_t$ is a (non-commutative) process of ± 1 valued variables with this covariance. Let us not that for a classical zero mean process, to have this covariance would require that $\xi_t = (\zeta e^{it} + \bar{\zeta}e^{-it})/2$ for all t where $\zeta = \xi + i\eta$ is a centred complex random variable with $\mathbb{E}(\xi^2) = \mathbb{E}(\eta^2) = 1$, $\mathbb{E}(\xi\eta) = 0$. In particular ξ_t has to take all values between $[-|\zeta|, |\zeta|]$. We see that in the quantum world discrete values and continuous time evolution is possible at the same time.

2 Heisenberg uncertainty principle

We let $m_\rho(A) = \mathrm{Tr}[\rho A]$ the average of A and $\mathrm{Var}_\rho(A) = m_\rho((A - m_\rho(A))^2)$ the variance of A in the state ρ.

Theorem 1. *Given an Hilbert space $\mathcal{H}$ (not necessarily finite dimensional) and bounded self–adjoint operators A, B it holds*

$$\mathrm{Var}_\rho(A)\mathrm{Var}_\rho(B) \geqslant \frac{1}{4}|m_\rho(i[A,B])|^2.$$

Proof. By replacing A, B with $A - m_\rho(A)$ and $B - m_\rho(B)$ is enough to consider mean zero operators. Then

$$m_\rho(i[A,B]) = i\,\mathrm{Tr}[\rho(AB - BA)] = -2\,\mathrm{Im}\sum_i \langle A\rho^{1/2} e_i, B\rho^{1/2} e_i\rangle$$

and then

$$|m_\rho(i[A,B])|^2 \leqslant 4\left[\sum_i \langle A\rho^{1/2} e_i, A\rho^{1/2} e_i\rangle\right]\left[\sum_i \langle B\rho^{1/2} e_i, B\rho^{1/2} e_i\rangle\right]$$

$$\leqslant 4\mathrm{Var}_\rho(A)\mathrm{Var}_\rho(B).$$

□

In particular if the operators do not commute there is no state which give zero dispersion to both at the same time. This fact encodes Heisenberg's uncertainty principle about the impossibility to measure with arbitrary precisions observables which interfere with each other. Non–commutativity is the mathematical tool which encodes this constraint.

3 Bell's inequalities

Consider four observables A_0, B_0, A_1, B_1 such that $[A_i, B_j] = 0$ for $i, j = 0, 1$ and all of them takes values ± 1. Let

$$R = A_0 B_0 + A_0 B_1 + A_1 B_0 - A_1 B_1$$

then

$$R^2 = 4 - [A_0, A_1][B_0, B_1].$$

In the classical case $[A_0,\, A_1] = [B_0,\, B_1] = 0$ so we have that $m_\rho(R) \leqslant 2$. In the quantum case $\|[A_0, A_1]\| = \|[B_0, B_1]\| \leqslant 2$ and we obtain *Tsirelson's bound*

$$m_\rho(R) \leqslant 2\sqrt{2}.$$

This bound is indeed tight and can be realised already on a tensor product of two Bernoulli spaces, showing that there is no classical assignment of probabilities to the four r.v. which can realise this particular correlation. In a different form this was originally obtained by Bell. The bound has also be saturated by an experiment carried over by Aspect et al.

The setup for the Bernoulli space is the following. Take the pure state

$$u = \frac{1}{\sqrt{2}}(|+_x\rangle \otimes |-_x\rangle - |-_x\rangle \otimes |+_x\rangle)$$

and $A_0 = \sigma_z \otimes \mathbb{I}$, $A_1 = \sigma_x \otimes \mathbb{I}$, $B_0 = \mathbb{I} \otimes \frac{(\sigma_z + \sigma_x)}{\sqrt{2}}$, $B_1 = \mathbb{I} \otimes \frac{(\sigma_z - \sigma_x)}{\sqrt{2}}$. We have

$$m_\rho(R) = 2\sqrt{2}.$$

4 Coherent states on the Bernoulli space

$\triangleright$**(creation and annihilation operators)** Introduce the further matrices

$$b^- = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \qquad b^+ = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \qquad b^\circ = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

called respectively annihilation, creation and preservation (or number) operator. Then

$$\mathbb{I} = b^- b^+ + b^+ b^-, \quad \sigma_x = b^+ + b^-, \qquad \sigma_y = i(b^+ - b^-), \qquad \sigma_z = b^- b^+ - b^+ b^- = \mathbb{I} - 2b^\circ.$$

note that $(b^\pm)^* = b^\mp$ and that $b^\circ = b^+ b^-$ is self–adjoint. In particular we have *canonical anticommutation relations*

$$\{b^+, b^-\} = b^- b^+ + b^+ b^- = \mathbb{I},$$

and *commutation relations*

$$[b^-, b^+] = b^- b^+ - b^+ b^- = \mathbb{I} - 2b^\circ.$$

We can construct also unitary operators. Let $z = x + iy = re^{i\varphi} \in \mathbb{C}$ and consider the skew–adjoint operator

$$J = zb^+ - \bar{z}b^- = x\,(b^+ - b^-) + iy\,(b^+ + b^-) = -i\,(x\sigma_x + y\sigma_y).$$

Now $J^2 = (zb^+ - \bar{z}b^-)^2 = -|z|^2\,(b^+ b^- + b^- b^+) = -|z|^2 = -r^2$.

so

$$W(z) = \exp(zb^+ - \bar{z}b^-) = \exp(J) = \sum_{n \geqslant 0} \frac{J^n}{n!} = \sum_{n \geqslant 0} \frac{J^{2n}}{(2n)!} + \sum_{n \geqslant 0} \frac{J^{2n+1}}{(2n+1)!}$$

$$= \sum_{n \geqslant 0} \frac{(-1)^n r^{2n}}{(2n)!} - i(\cos\varphi\,\sigma_x + \sin\varphi\,\sigma_y) \sum_{n \geqslant 0} \frac{(-1)^n r^{2n+1}}{(2n+1)!}$$

$$= \cos(r) - i(\cos\varphi\,\sigma_x + \sin\varphi\,\sigma_y)\sin(r) = \begin{pmatrix} \cos(r) & -ie^{-i\varphi}\sin(r) \\ -ie^{i\varphi}\sin(r) & \cos(r) \end{pmatrix}.$$

For every $z \in \mathbb{C}$ the family $(W(tz))_{t \in \mathbb{R}}$ is an unitary group. Discrete coherent vectors

$$\psi(z) = W(z) \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} \cos(r) \\ -ie^{-i\varphi}\sin(r) \end{pmatrix}$$

gives a different parametrization of unit vectors of $\mathcal{H}$. Is some sort of overcomplete continuous basis for the Hilbert space.

5 More than one quantum Bernoulli space, independently

▷ Consider now a family of N Bernoulli variables by taking the N-th tensor power $\mathcal{H}^N$ of the Hilbert space $\mathcal{H}$ together with a product state $\rho^N = \rho^{\otimes N}$ where $\rho \in \mathcal{S}(\mathcal{H})$. If A is a random variable on $\mathcal{H}$ we let $A^{(n)} = \tau^n(A) = \mathbb{I} \otimes \cdots \otimes A \otimes \cdots \otimes \mathbb{I}$ where the nontrivial factor is in the n-th position. For any $A, B \in \mathcal{L}(H)$, the random variables $A^{(n)}$ and $B^{(k)}$ commute if $n \neq k$ and their joint law is given by

$$\mathrm{Tr}_{\mathcal{H}^N}(\rho^N f(A^{(n)}) g(B^{(k)})) = \mathrm{Tr}_{\mathcal{H}}(\rho f(A)) \mathrm{Tr}_{\mathcal{H}}(\rho g(B)),$$

which is what we expect for independent random variables. This tensor product construction encodes classical independence in the non–commutative context. The family $(A^{(k)})_{k=1,\dots,N}$ is a classical random process of iid random variables.

▷ **The Quantum Bernoulli walk.** We introduce operators

$$X_n = \sum_{k=1}^{n} \sigma_x^{(k)}, Y_n = \sum_{k=1}^{n} \sigma_y^{(k)}, Z_n = \sum_{k=1}^{n} \sigma_z^{(k)}$$

for all $n = 1, \dots, N$ and observe that it holds

$$[X_n, Y_k] = 2i Z_{n \wedge k}, \qquad n, k = 1, \dots, N,$$

and cyclic permutations of this relation. On a product state ρ^N each of the processes $(X_n)_n$, $(Y_n)_n$, $(Z_n)_n$ are random walks with increments ± 1. On the other hand the process $(X_n, Y_n, Z_n)_{n=1,\dots,N}$ is a genuinely non–commutative object called the *quantum Bernoulli walk*.