Sheet 3

(Lectures of 24/4/2018 and 26/4/2018)

1 The Quantum Bernoulli walk and related processes

We continue in the setting of multiple independent Bernoulli spaces introduced last week.

▷ **Quantum Bernoulli walk.** We introduce operators

$$X_n = \sum_{k=1}^{n} \sigma_x^{(k)}, Y_n = \sum_{k=1}^{n} \sigma_y^{(k)}, Z_n = \sum_{k=1}^{n} \sigma_z^{(k)}$$

for all $n = 1, \ldots, N$ and observe that it holds

$$[X_n, Y_k] = 2i Z_{n \wedge k}, \qquad n, k = 1, \ldots, N,$$

and cyclic permutations of this relation. On a product state ρ^N each of the processes $(X_n)_n$, $(Y_n)_n$, $(Z_n)_n$ are random walks with increments ± 1. On the other hand the process $(X_n, Y_n, Z_n)_{n=1,\ldots,N}$ is a genuinely non–commutative object called the *quantum Bernoulli walk*.

▷ **Spectra.** We have already seen that the Pauli matrices are related to rotations. This motivates us to consider the operator $R_n^2 = X_n^2 + Y_n^2 + Z_n^2$ and observe that if $k \geqslant n$ we have

$$[Z_k, X_n^2 + Y_n^2 + Z_n^2] = X_n[Z_k, X_n] + [Z_k, X_n]X_n + Y_n[Z_k, Y_n] + [Z_k, Y_n]Y_n$$

$$= 2i(X_n Y_{n\wedge k} + Y_{n\wedge k} X_n - Y_n X_{n\wedge k} - X_{n\wedge k} Y_n) = 0$$

and similarly $[X_k, X_n^2 + Y_n^2 + Z_n^2] = [Y_k, X_n^2 + Y_n^2 + Z_n^2] = 0$.

Moreover we introduce collective operators

$$B_n^{\pm} = \sum_{k=1}^{n} (b^{\pm})^{(k)} = \frac{(X_n \mp i Y_n)}{2}, \qquad B_n^{\circ} = \sum_{k=1}^{n} (b^{\circ})^{(k)} = \frac{n - Z_n}{2},$$

which also commute with $X_n^2 + Y_n^2 + Z_n^2$. On the other hand we have

$$[B_n^-, B_n^+] = \sum_{k=1}^{n} [(b^-)^{(k)}, (b^+)^{(k)}] = n - 2B_n^{\circ} = Z_n,$$

$$[B_n^+, B_n^{\circ}] = \sum_{k=1}^{n} [(b^+)^{(k)}, (b^+)^{(k)}(b^-)^{(k)}] = \sum_{k=1}^{n} (b^+)^{(k)} = B_n^+,$$

and $[B_n^-, B_n^{\circ}] = -B_n^-$. These relations imply that if φ_k is an eigenvector of B_n° with eigenvalue $k \in \sigma(B_n^{\circ}) = \{0, \ldots, n\}$ then

$$B_n^{\circ}(B_n^{\pm}\varphi_k) = B_n^{\pm} B_n^{\circ}\varphi_k \pm B_n^{\pm}\varphi_k = (k \pm 1) B_n^{\pm}\varphi_k$$

so either $B_n^{\pm}\varphi_k = 0$ or $B_n^{\pm}\varphi_k$ is another eigenvector with eigenvalue $k \pm 1$. Since B_n° commutes with R_n we can chooses common eigenvectors. Note that at this point is not clear what is the spectrum of R_n. However we can observe that

$$4B_n^+ B_n^- = (X_n - iY_n)(X_n + iY_n) = X_n^2 - iY_n X_n + iX_n Y_n + Y_n^2 = R_n^2 - Z_n^2 - 2Z_n.$$

Let $\varphi_{r,a}$ be a joint eigenvector of R_n, Z_n such that $B_n^- \varphi_{r,a} = 0$. This always exists since the Hilbert space is finite dimensional and repeated application of B_n^- produces orthogonal eigenvectors of Z_n, R_n unless it gives 0. Then we must have

$$r^2 - a^2 - 2a = 0.$$

On the other hand, from this vector we can repeatedly apply B_n^+ until we get zero and generate a sequence of common eigenvectors of R_n and Z_n which we denote by $\varphi_{r,a-2k} = (B_n^+)^k \varphi_{r,a}$ since $Z_n = n - 2B_n^{\circ}$ so from the equality $4B_n^- B_n^+ = 4B_n^+ B_n^- + 4Z_n = R_n^2 - Z_n^2 + 2Z_n$ applied to $\varphi_{r,a-2k}$ we obtain also the relation

$$0 = r^2 - (a-2k)^2 + 2(a-2k) = r^2 - a^2 + 4ka - 4k^2 + 2a - 4k$$

which together imply

$$4a(k+1) = 4k^2 + 4k = 4k(k+1).$$

Since $k > 0$ by assumption we must have $k = a$ and $r^2 = k^2 + 2k = k(k+2)$ for some $k = 1, 2, \ldots$ It follows that the spectrum of R_n has to belong to the set $\{r : r^2 = k(k+2) \text{ for } \quad k \in \{1, 2, \ldots\}\}$.

In order to get nicer formulas, more in line with traditional physical notations, we can then introduce a new observable J_n (which is a function of R_n) such that

$$J_n(J_n + 1) = \frac{X_n^2 + Y_n^2 + Z_n^2}{4}$$

in such a way that the spectrum of J_n is in the set $\mathbb{N}/2$. So the value of k above coincides with $2j$. Therefore if $j \in \mathbb{N}/2$ is in the spectrum of J_n then there exists a set of $2j+1$ common eigenvectors of J_n and $M_n = Z_n/2$ such that the operator M_n takes on them the values $\{-j, -j+1, \ldots, 0, \ldots, j-1, j\}$. Inversely, by the above construction, if there exists an eigenvector of M_n with eigenvalue m then there must exists a full set of $2j+1$ common eigenvectors (a *ladder*) of J_n and M_n with $J_n \geqslant |m|$. We denote the vectors of this ladder with $\psi_{j,m}$. We are going to choose a convenient normalization so for the moment these vectors are defined by the relations

$$B_n^+ \psi_{j,m} = \psi_{j,m-1}.$$

To determine completely the structure of the ladder space we need to fix the action of the annihilation operators. Observe that

$$B_n^+ B_n^- = J_n(J_n + 1) - M_n(M_n + 1) = (J_n - M_n)(J_n + M_n + 1)$$

and

$$B_n^- B_n^+ = B_n^+ B_n^- + 2M_n = J_n(J_n + 1) - M_n(M_n - 1) = (J_n + M_n)(J_n - M_n + 1)$$

so

$$B_n^- \psi_{j,m} = B_n^- B_n^+ \psi_{j,m+1} = (j + m + 1)(j - m)\psi_{j,m+1}.$$

A convenient normalization which makes nicer formulas for $B_n^{\pm}$ is to take vectors

$$\theta_{j,m} = \psi_{j,m} \prod_{m' \in \{m, m+1, \ldots, j-1\}} (j - m' + 1)^{-1}$$

so

$$B_n^- \theta_{j,m} = \frac{B_n^- \psi_{j,m}}{\prod_{m' \in \{m, m+1, \ldots, j-1\}} (j + m' + 1)} = (j - m)\theta_{j,m+1}.$$

and

$$B_n^+ \theta_{j,m} = \frac{B_n^+ \psi_{j,m}}{\prod_{m' \in \{m, m+1, \ldots, j-1\}} (j + m' + 1)} = (j + m)\theta_{j,m-1}.$$

Moreover

$$\langle \theta_{j,m-1}, \theta_{j,m-1} \rangle = \frac{\langle B_n^+ \theta_{j,m}, B_n^+ \theta_{j,m} \rangle}{(j+m)^2} = \frac{\langle \theta_{j,m}, B_n^- B_n^+ \theta_{j,m} \rangle}{(j+m)^2} = \frac{(j - m + 1)}{(j+m)} \langle \theta_{j,m}, \theta_{j,m} \rangle$$

and fixing $\langle \theta_{j,-j}, \theta_{j,-j} \rangle = 1$ we have

$$\langle \theta_{j,m}, \theta_{j,m} \rangle = \prod_{m'=-j+1}^{m} \frac{j + m'}{j - m' + 1} = \prod_{k=1}^{m+j} \frac{k}{2j + 1 - k} = \frac{(j+m)!(j-m)!}{(2j)!}$$

for all $m = -j + 1, \ldots, j$.

$\triangleright$ In our Bernoulli setting there exists surely eigenvectors of M_n with eigenvalue $m \in \{-n/2, -n/2+1, \ldots, n/2-1, n/2\}$ which therefore implies existence of a ladder with $j = n/2$ for J_n. The dimension of the vector space generated by this ladder is $2j + 1$ so it does not exhaust the full dimension of the space $\mathcal{H}^N$ which is 2^N.

$\triangleright$ **Products of irreps.** To understand how $\mathcal{H}^N$ decomposes into ladder spaces we proceed by induction. Consider a subspace E_j^n of $\mathcal{H}^n$ generated by a ladder with $J_n = j$, this space if of dimension $2j+1$. The random variable M_{n+1} on $E_j^n \otimes \mathcal{H} \subseteq \mathcal{H}^{n+1}$ can take largest value $j + 1/2$ so in $E_j^n \otimes \mathcal{H}$ there should be a ladder $E_{j+1/2}^{n+1}$. The full spectrum of M_{n+1} is $\{-j - 1/2, \ldots, j + 1/2\}$ and the multiplicity of each eigenvalue is given by the ways of decomposing it into the sum of an element of $\{-j/2, \ldots, j/2\}$ and $\{-1/2, 1/2\}$. From the consideration of this multiplicity we can deduce that there are at most two laddes $E_{j+1/2}^{n+1}$ and $E_{j'}^{n+1}$ in the decomposition of $E_j^n \otimes \mathcal{H}$, namely $E_j^n \otimes \mathcal{H} = E_{j'}^{n+1} \oplus E_{j+1/2}^{n+1}$. In order to determine j' we consider the dimensions of these spaces:

$$2(2j+1) = [2(j+1/2)+1] + [2j'+1]$$

which implies that $j' = j - 1/2$. So we have found

$$E_j^n \otimes \mathcal{H} = E_{j-1/2}^{n+1} \oplus E_{j+1/2}^{n+1}.$$

$\triangleright$ **General case.** This is a particular case of a general theorem about products of irreducible representations of the relations $[X_n, Y_n] = 2i Z_n$ (and cyclic permutations). The latter generate the Lie algebra of the group SU(2). The irreducible representations are labeled by $j = 0, 1/2, 1, 3/2, \ldots$ and the general addition theorem says that the product of an irreducible representation labeled by j_1 and another with j_2 can be decomposed in a direct sum of irreducible representations indexed by each $j = |j_1 - j_2|, \ldots, j_1 + j_2$ taken with multiplicity one. This is related to the relation

$$(2j_1+1)(2j_2+1) = \sum_{j'=|j_1-j_2|}^{j_1+j_2} (2j'+1).$$

▷ **A commutative process of spins.** We consider now the random variables $S_n = 2J_n + 1$ which are associated to the dimension of the irreducible representations we constructed above. By definition S_n commutes with X_n, Y_n, Z_n (because R_n does) and commutes also with X_k, Y_k, Z_k for $k > n$ (why?). So it commutes with S_k for $k > n$. In particular the process $(S_n)_{n=1,\dots,N}$ is a classical stochastic process which we call the *total spin process*. Note that S_n commutes with Z_n but not with Z_k for $k < n$.

A common eigenvector ψ_s for $(S_n)_{n=1,\dots,N} = (2j_n + 1)_{n=1,\dots,N}$ must belong to a space of the form $E^n_{j_n} \otimes \mathcal{H}^{N-n}$ for all $n = 1, \dots, N$. In particular it belongs to a ladder space $E^N_{j_N}$ and so it is completely determined as soon as we know the value of M_N on it. Note that M_N commutes with all $(S_n)_{n=1,\dots,N}$ so can be determined together with the path of this particular process. We conclude that the projector Π_s on $(S_n)_{n=1,\dots,N} = (s_n)_{n=1,\dots,N}$ has dimension at least s_N. We want now to argue that the dimension is exactly s_N, namely that there cannot be two ladders for J_N with all the same values of $J_1, \dots, J_{N-1}$. Indeed for the product rule of representations we already know that we have the relation

$$E^k_{j_k} \otimes \mathcal{H}^{N-k} = (E^{k+1}_{j_k+1/2} \otimes \mathcal{H}^{N-k-1}) \oplus (E^{k+1}_{j_k-1/2} \otimes \mathcal{H}^{N-k-1})$$

given ladder $E^k_{j_k} \otimes \mathcal{H}^{N-k}$ at step k splits into exactly two other ladders with different values of J_{k+1}. Since at the initial time $(k=1)$ there is only one space of the form $E^k_{j_k} \otimes \mathcal{H}^{N-k}$ it follows by induction that for each $k = 1, \dots, N$ there is only one space of the form $E^k_{j_k} \otimes \mathcal{H}^{N-k}$ in which the operators $J_1, \dots, J_{k-1}$ takes the fixed values $j_1, \dots, j_{k-1}$. We conclude that there is only one allowed ladder $E^N_{j_N}$ in which all $J_1, \dots, J_N$ take values $j_1, \dots, j_N$ and that the dimension of the spectral projector Π_s is s_N.

As a consequence, if $\rho^N = \mathbb{I}/2^N$ is the tracial state we have

$$\mathbb{P}(S_n = s_n : n = 1, \dots, N) = \mathrm{Tr}_{\mathcal{H}^N}[\rho^N \Pi_s] = \frac{s_N}{2^N}.$$

That is we have determined the complete law of this commutative process. It is now easy to find out that this is a Markov process for which $S_1 = 2$ and

$$\mathbb{P}(S_{k+1} = s_k \pm 1 | S_k = s_k) = \frac{s_k \pm 1}{2 s_k}.$$

This is a discrete analog of the 3–dimensional Bessel process.