Sheet 4

(Lecture of 8/5/2018)

1 A central limit theorem of DeMoivre–Laplace type

▷ **Gauss and Poisson.** We observe now that for all $\theta \in \mathbb{R}$ the random variable

$$e^{i\theta}b^{+}+e^{-i\theta}b^{-},$$

has a symmetric law on $\{\pm 1\}$ in the state $|e_0\rangle\langle e_0|$. By the classical CLT the random variable

$$G_N=\frac{e^{i\theta}B_N^{+}+e^{-i\theta}B_N^{-}}{\sqrt{N}},$$

in the product state $\rho^N=(|e_0\rangle\langle e_0|)^{\otimes N}$ converges towards a centred Gaussian random variable.

Similarly, the random variable $b^{+}b^{-}$ has law δ_0 and therefore

$$Z_N=B_N^{\circ}$$

(without any normalization) converges to the constant 0.

Finally, the random variable

$$\left(1+\frac{|z|^2}{N}\right)^{-1}\left[\left(1-\frac{|z|^2}{N}\right)b^{+}b^{-}+\frac{z}{\sqrt{N}}b^{+}+\frac{\bar{z}}{\sqrt{N}}b^{-}+\frac{|z|^2}{N}\mathbb{I}\right]=\left(1+\frac{|z|^2}{N}\right)^{-1}\begin{pmatrix}\frac{|z|^2}{N} & \frac{\bar{z}}{\sqrt{N}}\\ \frac{z}{\sqrt{N}} & 1\end{pmatrix}$$

has spectrum $\{0,1\}$ with respective probabilities $\left(1+\frac{|z|^2}{N}\right)^{-1}$ and $\frac{|z|^2}{N}\left(1+\frac{|z|^2}{N}\right)^{-1}$ on the pure state $|e_0\rangle\langle e_0|$. As a consequence, on the product state $\rho^N=(|e_0\rangle\langle e_0|)^{\otimes N}$, the random variable

$$\begin{aligned}P_N&=\left(1+\frac{|z|^2}{N}\right)^{-1}\left[\left(1-\frac{|z|^2}{N}\right)B_N^{\circ}+\frac{z}{\sqrt{N}}B_N^{+}+\frac{\bar{z}}{\sqrt{N}}B_N^{-}+|z|^2\mathbb{I}\right]\\&=\left(1+\frac{|z|^2}{N}\right)^{-1}\left[\left(1-\frac{|z|^2}{N}\right)Z_N+G_N^{z}+|z|^2\mathbb{I}\right]\end{aligned}$$

where

$$G_N^{z}=\frac{z}{\sqrt{N}}B_N^{+}+\frac{\bar{z}}{\sqrt{N}}B_N^{-},$$

has a Binomial law with parameters $\left(N,\frac{|z|^2}{N}\left(1+\frac{|z|^2}{N}\right)^{-1}\right)$ which converges to a Poisson random variable with parameter $|z|^2$ as $N\to\infty$. Note that P_N is the linear combination of a random variable with constant law

$$\left(1+\frac{|z|^2}{N}\right)^{-1}\left[\left(1-\frac{|z|^2}{N}\right)Z_N+|z|^2\mathbb{I}\right]$$

and of a random variable G_N^z which converges to a Gaussian.

▷ **A first computation.** From these two observation become interesting to study the convergence of the triple of non-commutative (non-self-adjoint) operators $\big(B_N^\circ, B_N^+/\sqrt{N}, B_N^-/\sqrt{N}\big)$ as $N\to\infty$ in the state $(|e_0\rangle\langle e_0|)^{\otimes N}$. Since we lack of a proper notion of law for such a triplet we will restrict (in a first moment) to investigate the limit of moments of these operators and look at averages of the form

$$\big\langle \varphi_0, P\big(B_N^\circ, B_N^+/\sqrt{N}, B_N^-/\sqrt{N}\big)\varphi_0\big\rangle \tag{1}$$

where P is a generic non-commutative polynomial in three variables and $\varphi_0 = e_0^{\otimes N}$. Now observe that $Z_N\varphi_0 = N\varphi_0$ so starting from φ_0 we generate a ladder with $j = N/2$ since no larger values of Z_N are possible in $\mathcal{H}^N$. We will label the vectors of this ladder with $(\varphi_n)_{n=0,\ldots,N}$ such that

$$B_N^\circ\varphi_n = n\varphi_n, \qquad M_N\varphi_n = \frac{Z_N}{2}\varphi_n = \frac{N-2B_N^\circ}{2}\varphi_n = \left(\frac{N}{2}-n\right)\varphi_n.$$

Therefore we must have

$$B_N^+\varphi_n = \alpha_n\varphi_{n+1}, \qquad B_N^-\varphi_n = \beta_n\varphi_{n-1},$$

and we need to determine the coefficients α_n, β_n. We need to have

$$n(N-n+1)\varphi_n = (N/2-M_n)(N/2+M_n+1)\varphi_n = B_N^+B_N^-\varphi_n = \alpha_{n-1}\beta_n\varphi_n$$

so we can take $\alpha_n = \sqrt{(n+1)(N-n)}$ and $\beta_n = \sqrt{n(N-n+1)}$. With this choice we have

$$\langle\varphi_n,\varphi_n\rangle = \frac{1}{\alpha_{n-1}^2}\langle B_N^+\varphi_{n-1}, B_N^+\varphi_{n-1}\rangle = \frac{1}{\alpha_{n-1}^2}\langle\varphi_{n-1}, B_N^-B_N^+\varphi_{n-1}\rangle = \langle\varphi_{n-1},\varphi_{n-1}\rangle = \cdots = \langle\varphi_0,\varphi_0\rangle = 1$$

since $\langle\varphi_0,\varphi_0\rangle = 1$. Now

$$B_N^\circ\varphi_n = n\varphi_n, \quad \frac{B_N^+}{\sqrt{N}}\varphi_n = \sqrt{(n+1)(1-n/N)}\,\varphi_{n+1}, \qquad \frac{B_N^-}{\sqrt{N}}\varphi_n = \sqrt{n(1-(n-1)/N)}\,\varphi_{n-1}.$$

If P is a polynomial of degree d only vectors φ_k with $k\leqslant d$ are involved in the expression (1). It is then easy to deduce that

$$\lim_N \big\langle \varphi_0, P\big(B_N^\circ, B_N^+/\sqrt{N}, B_N^-/\sqrt{N}\big)\varphi_0\big\rangle \to \langle\psi_0, P(a^\circ, a^+, a^-)\psi_0\rangle$$

where $(\psi_n)_{n\geqslant 0}$ is a family of ortogonal vectors of an Hilbert space $\mathcal{F}$ (apriori unrelated to $\mathcal{H}^N$) and (a°, a^+, a^-) are operators on $\mathcal{F}$ defined by

$$a^\circ\psi_n = n\psi_n, \quad a^+\psi_n = \sqrt{(n+1)}\,\psi_{n+1}, \quad a^-\psi_n = \sqrt{n}\,\psi_{n-1},$$

Note that if we assume that $\langle\psi_0,\psi_0\rangle = 1$ we have the normalization condition

$$\langle\psi_n,\psi_n\rangle = \frac{\langle a^+\psi_{n-1}, a^+\psi_{n-1}\rangle}{n} = \frac{\langle e_{n-1}, \psi^-\psi^+e_{n-1}\rangle}{n} = \langle\psi_{n-1},\psi_{n-1}\rangle = \langle\psi_0,\psi_0\rangle = 1$$

and the relations

$$a^+a^-\psi_n = n\psi_n, \qquad a^-a^+\psi_n = (n+1)\psi_n,$$

so in particular

$$[a^+, a^-]\psi_n = (a^+a^- - a^-a^+)\psi_n = \psi_n$$

for all $n \geqslant 0$.

$\triangleright$ By the above considerations we already know that the operator

$$G_\infty = e^{i\theta}a^+ + e^{-i\theta}a^-,$$

should have a Gaussian law. The fact that it is self-adjoint is now nontrivial since this operator is not bounded. Similarly, the operator

$$Z_\infty = a^\circ + |z|^2\mathbb{I}$$

has law δ_0 on $|\varphi_0\rangle\langle\varphi_0|$ and

$$P_\infty = a^\circ + za^+ + \bar{z}\,a^- + |z|^2\mathbb{I},$$

should have a Poisson law of parameter $|z|^2$, but again this would require justification since these operator is also not bounded.

$\triangleright$ Note also that in the non-commutative world a Poisson random variable can be obtained by summing a Gaussian G_∞ with a random variable Z_∞ with a constant law. Of course the point is that G_∞ does not commute with Z_∞.

2 Quasi–characteristic functions and the quantum Gaussian

$\triangleright$ In order to explore more in detail the above convergence and its generalisation into full–fledged non-commutative central limit theorem we investigate the general problem of convergence of quasi–characteristic functions of arbitrary non–commuting operators. For simplicity we continue to stick to a finite dimensional context, which however we will be force to leave soon, since as we seen above the definition of the Gaussian requires unbounded operators (because the support of the Gaussian law is all $\mathbb{R}$) and therefore an infinite dimensional Hilbert space.

$\triangleright$ In this section we consider an arbitrary finite dimensional quantum probability space $(\mathcal{H}, \rho)$ where $\mathcal{H}$ is a finite dimensional Hilbert space with a state ρ and a family of self–adjoint operators $(A_j)_j \subseteq \mathcal{L}(\mathcal{H})$. We construct N independent copies of this quantum probability space via tensorization and operators $A_j^{(k)}$ operating on the k-th copy. Finally for any A_j we consider the operator $\sigma_N(A_j)$ defined by

$$\sigma_N(A) = \frac{1}{\sqrt{N}}\sum_{k=1}^{n} [A^{(k)} - \rho(A)].$$

Theorem 1. *For all $\alpha_1, \ldots, \alpha_n \in \mathbb{C}$ we have*

$$\lim_{N\to\infty} \rho^N\left(\prod_{k=1}^{n} e^{i\alpha_k\sigma_N(A_k)}\right) = \exp\left(-\frac{1}{2}\sum_{1\leqslant j,k\leqslant n} Q(A_j, A_k)\alpha_j\alpha_k - i\sum_{1\leqslant j<k\leqslant n} \kappa(A_j, A_k)\alpha_j\alpha_k\right) = \Phi(\alpha)$$

where $Q(A, B) = \mathrm{Re}\,(\rho(AB) - \rho(A)\rho(B))$ and $\kappa(A, B) = \mathrm{Im}\,\rho(AB) = -i\rho([A, B])/2$.

Proof. We can assume that $\rho(A_j) = 0$ for all j. Now note that

$$\rho^N\left(\prod_{k=1}^{n} e^{i\alpha_k\sigma_N(A_k)}\right) = \left[\rho\left(\prod_{k=1}^{n} e^{iN^{-1/2}\alpha_kA_k}\right)\right]^N$$

and using the Taylor expansion

$$e^{iN^{-1/2}\alpha_k A_k} = 1 + iN^{-1/2}\alpha_k A_k + \frac{(iN^{-1/2}\alpha_k A_k)^2}{2} + \frac{(iN^{-1/2}\alpha_k A_k)^3}{3!}\int_0^1 e^{iN^{-1/2}\alpha_k \tau_k A_k}\lambda(\mathrm{d}\tau_k)$$

we have

$$\rho\left(\prod_{k=1}^n e^{iN^{-1/2}\alpha_k A_k}\right) = 1 - \sum_k \frac{\rho[(\alpha_k A_k)^2]}{2N} - \frac{1}{N}\sum_{k<j} \rho[\alpha_k A_k \alpha_j A_j] + O\left(\frac{\sup_j \|A_j\alpha_j\|^3 e^{N^{-1/2}|\alpha_j|\|A_j\|}}{N^{3/2}}\right)$$

therefore, pointwise in α and uniformly in compacts

$$\left[\rho\left(\prod_{k=1}^n e^{iN^{-1/2}\alpha_k A_k}\right)\right]^N \to \exp\left(-\sum_k \frac{\rho[(\alpha_k A_k)^2]}{2} - \sum_{k<j} \rho[\alpha_k A_k \alpha_j A_j]\right).$$

Now it suffices to note that

$$\begin{aligned} &-\frac{1}{2}\sum_{1\leqslant j,k\leqslant n} Q(A_j, A_k)\alpha_j\alpha_k - i\sum_{1\leqslant j<k\leqslant n} \kappa(A_j, A_k)\alpha_j\alpha_k \\ &= -\frac{1}{2}\sum_{1\leqslant j,k\leqslant n} \mathrm{Re}\,\rho(A_j A_k)\alpha_j\alpha_k - i\sum_{1\leqslant j<k\leqslant n} \mathrm{Im}\,\rho(A_j A_k)\alpha_j\alpha_k \\ &= -\frac{1}{2}\sum_{1\leqslant j\leqslant n} \rho(A_j^2)\alpha_j^2 - \sum_{1\leqslant j<k\leqslant n} \rho(A_j A_k)\alpha_j\alpha_k \end{aligned}$$

proving the claim. □

▷ This theorem suggests that the function $\Phi(\alpha)$ should be the quasi-characteristic function of a quantum probability space $(\mathcal{F}, \omega)$ endowed with a family of non-commuting (why?) operators $\varphi(A_j)$ labeled by the A_j such that, on the state ω we have

$$\omega\left(\prod_{k=1}^n e^{i\alpha_k\varphi(A_k)}\right) = \exp\left(-\frac{1}{2}\sum_{1\leqslant j,k\leqslant n} Q(A_j, A_k)\alpha_j\alpha_k - i\sum_{1\leqslant j<k\leqslant n} \kappa(A_j, A_k)\alpha_j\alpha_k\right).$$

In classical probability the existence of such a probabability, with prescribed characteristic function, would be automatic given the (uniform) convergence of the characteristic functions itself. Here it is not so simple. But the information we possess will be enough to show explicitly the existence of such a quantum probability space which is labeled by the functions Q, κ.

▷ If we recall the considerations of the previous section and apply the last theorem to the operators $\sigma_x, \sigma_y, \sigma_z$ on the quantum Bernoulli space on the pure state e_0 we have that

$$Q(\sigma_a, \sigma_b) = \mathrm{Re}\,\langle e_0|\sigma_a\sigma_b|e_0\rangle = \delta_{a,b}, \qquad \kappa(\sigma_x, \sigma_y) = \frac{1}{2i}\langle e_0|[\sigma_x, \sigma_y]|e_0\rangle = 1,$$

so in this case for example

$$\omega(e^{i\alpha_x\varphi(\sigma_x)}e^{i\alpha_y\varphi(\sigma_y)}) = \exp\left(-\frac{1}{2}(\alpha_x^2 + \alpha_y^2) - i\alpha_x\alpha_y\right).$$