# Sheet 5

(Lectures of 15/5/2018 and 17/5/2018)

Recall the theorem we proved last week:

**Theorem 1.** *For all $\alpha_1, \ldots, \alpha_n \in \mathbb{C}$ we have*

$$\lim_{N \to \infty} \rho^N \left( \prod_{k=1}^{n} e^{i\alpha_k \sigma_N(A_k)} \right) = \Phi_A(\alpha)$$

*with*

$$\Phi_A(\alpha) = \exp\left( -\frac{1}{2} \sum_{1 \leqslant j,k \leqslant n} Q(A_j, A_k)\alpha_j\alpha_k - i \sum_{1 \leqslant j<k \leqslant n} \kappa(A_j, A_k)\alpha_j\alpha_k \right)$$

*where $Q(A, B) = \mathrm{Re}\,(\rho(AB) - \rho(A)\rho(B))$ and $\kappa(A, B) = \mathrm{Im}\,\rho(AB) = -i\rho([A, B])/2$.*

▷ Theorem 1 suggests that the function $\Phi_A(\alpha)$ should be the quasi-characteristic function of a quantum probability space $(\mathcal{F}, \omega)$ endowed with a family of non-commuting (why?) operators $\varphi(A_j)$ labeled by the $A_j$ such that, on the state $\omega$ we have

$$\omega\left( \prod_{k=1}^{n} e^{i\alpha_k \varphi(A_k)} \right) = \exp\left( -\frac{1}{2} \sum_{1 \leqslant j,k \leqslant n} Q(A_j, A_k)\alpha_j\alpha_k - i \sum_{1 \leqslant j<k \leqslant n} \kappa(A_j, A_k)\alpha_j\alpha_k \right).$$

In classical probability the existence of such a probability, with prescribed characteristic function, would be automatic given the (uniform) convergence of the characteristic functions itself. Here it is not so simple. But the information we possess will be enough to show explicitly the existence of such a quantum probability space which is labeled by the functions $Q, \kappa$.

▷ If we recall the considerations of the previous section and apply the last theorem to the operators $\sigma_x, \sigma_y, \sigma_z$ on the quantum Bernoulli space on the pure state $e_0$ we have that

$$Q(\sigma_a, \sigma_b) = \mathrm{Re}\,\langle e_0|\sigma_a\sigma_b|e_0\rangle = \delta_{a,b}, \qquad \kappa(\sigma_x, \sigma_y) = \frac{1}{2i}\langle e_0|[\sigma_x, \sigma_y]|e_0\rangle = 1,$$

so in this case for example

$$\omega(e^{i\alpha_x\varphi(\sigma_x)} e^{i\alpha_y\varphi(\sigma_y)}) = \exp\left( -\frac{1}{2}(\alpha_x^2 + \alpha_y^2) - i\alpha_x\alpha_y \right).$$

▷ If $\kappa = 0$ this corresponds to the caracteristic function of a family of Gaussian variables.

▷ On the other hand if $\kappa \neq 0$ we need necessarily have $Q \neq 0$, indeed the Hermitian form $L(A, B) = \rho((A - \rho(A))(B - \rho(B)))$ satisfy the Cauchy–Schwartz inequality

$$\kappa(A, B)^2 \leqslant |L(A^*, B)|^2 \leqslant L(A^*, A)L(B^*, B) = Q(A, A)Q(B, B)$$

which is a restatement of Heisenberg's uncertainty principle.

## 1 A quantum Gaussian white noise

We give here an spatial extension of Theorem 1. We imagine that copies of the Hilbert space $\mathcal{H}$ are indexed by elements of a finite box $\Lambda_{L,N} = [-LN, LN]^d \subseteq \mathbb{Z}^d$ in $d$ dimensions. Here $L, N \in \mathbb{N}$ are two integers. With this additional structure we can consider averaging operators indexed by smooth compactly supported functions $\varphi \in \mathcal{S}(\mathbb{R}^d)$ supported on $[-L, L]^d \subseteq \mathbb{R}^d$ and let

$$\sigma_N(A, \varphi) = \sum_{k \in \Lambda_{L,N}} \varphi(k/N) A^{(k)}$$

in $\mathcal{H}^{\Lambda_{L,N}}$. Then, essentially using the same arguments as in the proof of Theorem 1, we can verify that

**Theorem 2.** *For all $\alpha_1, \ldots, \alpha_n \in \mathbb{C}$ and $\varphi_1, \ldots, \varphi_k \in \mathcal{S}(\mathbb{R}^d)$ with compact support we have*

$$\lim_{L \to \infty} \lim_{N \to \infty} \rho^{\Lambda_{L,N}} \left( \prod_{k=1}^{n} e^{i\alpha_k \sigma_N(A, \varphi)} \right) = \Phi_A(\alpha\varphi)$$

$$\Phi_A(\alpha\varphi) = \exp\left( -\frac{1}{2} \sum_{1 \leqslant j,k \leqslant n} Q(A_j, A_k) \langle\langle \varphi_j, \varphi_k \rangle\rangle \alpha_j \alpha_k - i \sum_{1 \leqslant j < k \leqslant n} \kappa(A_j, A_k) \langle\langle \varphi_j, \varphi_k \rangle\rangle \alpha_j \alpha_k \right)$$

*where $Q(A, B) = \mathrm{Re}\,(\rho(AB) - \rho(A)\rho(B))$, $\kappa(A, B) = \mathrm{Im}\,\rho(AB) = -i\rho([A, B])/2$ and*

$$\langle\langle \varphi_j, \varphi_k \rangle\rangle = \int_{\mathbb{R}^d} \varphi_j(x) \varphi_k(x) \mathrm{d}x.$$

▷ Theorem 2 suggests that the function $\Phi_A(\varphi)$ should be the quasi-characteristic function of a quantum probability space $(\mathcal{F}, \omega)$ endowed with a family of non-commuting (why?) operators $\psi(A_j)(\varphi)$ labeled by the $A_j$ and $\varphi$ such that, on the state $\omega$ we have

$$\omega\left( \prod_{k=1}^{n} e^{i\alpha_k \psi(A_k)(\varphi)} \right) = \Phi_A(\alpha\varphi).$$

In classical probability the existence of such a probabability, with prescribed characteristic function, would be automatic given the (uniform) convergence of the characteristic functions itself. Here it is not so simple. But the information we possess will be enough to show explicitly the existence of such a quantum probability space which is labeled by the functions $Q, \kappa$.

▷ Note that when $\kappa = 0$ the function $\Phi_A(\alpha\varphi)$ is the characteristic function of a finite dimensional projection of a vector valued white noise $\xi$ over $\mathbb{R}^d$, that is, a vector valued, centered Gaussian process $\xi$ indexed by $\varphi \in \mathcal{S}(\mathbb{R}^d)$ with covariance

$$\mathbb{E}[\xi_i(\varphi) \xi_j(\psi)] = Q(A_i, A_j) \langle\langle \varphi, \psi \rangle\rangle.$$

## 2 Existence of the quantum Gaussian

▷Here we want to explicitly construct the space $(\mathcal{F}, \omega)$. Consider the polynomial algebra $\mathcal{P}$ generated by $(U(A))_{A \in \mathcal{A}}$ wheer $\mathcal{A}$ is the vector space of (bounded) selfadjoint operators on $\mathcal{H}$ and endow it with the inner product

$$\left\langle \prod_j U(A'_j), \prod_k U(A_k) \right\rangle = \lim_N \rho^N \left[ \left( \prod_j e^{i\sigma_N(A_j)} \right)^* \prod_k e^{i\sigma_N(A_k)} \right]$$

extended by linearity.

By Cauchy–Schwartz inequality the elements $a\in\mathcal{P}$ such that $\langle a,a\rangle=0$ form a vector subspace $\mathcal{I}\subseteq\mathcal{P}$. As vector spaces the quotient space $\mathcal{P}/\mathcal{I}$ is an inner product space which can be completed into an Hilbert space $\mathcal{F}$.

Now if $b\in\mathcal{P}$ and $a\in\mathcal{I}$ we have $ba=\sum_r c_r b_r a$ where $b_r$ are monomials of the form $\prod_k U(A_k)$. Therefore if we denote by $b_{r,N}$ the finite $N$ representative of $b_r$ and $a_N$ that of $a$ we have

$$\|b_r a\|^2=\lim_N \rho^N[a_N^* b_{r,N}^* b_{r,N} a_N]\leqslant \lim_N \|b_{r,N}\|^2\rho^N[a_N^* a_N]\leqslant \lim_N \rho^N[a_N^* a_N]=\|a\|^2$$

where we used that $\|b_{r,N}\|=1$ which implies the inequality $\mathbb{I}-b_{r,N}^* b_{r,N}\geqslant 0$ and which in turns implies that $a_N^*(\mathbb{I}-b_{r,N}^* b_{r,N})a_N\geqslant 0$. We conclude that $\mathcal{I}$ is a left ideal of $\mathcal{P}$.

Now $U(A_k)$ acts on this Hilbert space naturally as left multiplication

$$U(A_k)\left(\sum_r c_r\left[\prod_j U(A_{r,j})\right]+\mathcal{I}\right)=\sum_r c_r\left[U(A_k)\prod_j U(A_{r,j})\right]+\mathcal{I},$$

and we have for all $b\in\mathcal{P}$, $\|U(A_k)b\|^2=\lim_N\rho^N[\Phi_N^*(e^{i\sigma_N(A_k)})^* e^{i\sigma_N(A_k)}\Phi_N]$, where $\Phi_N$ is the finite $N$ approximation of $b$. Therefore since $e^{i\sigma_N(A_k)}$ is unitary $(e^{i\sigma_N(A_k)})^* e^{i\sigma_N(A_k)}=1$ and $\|U(A_k)b\|^2=\|b\|^2$, and showing that $U(A_k)$ is a bounded operator on $\mathcal{P}/\mathcal{I}$ which can be extended uniquely by continuity to $\mathcal{F}$. One sees also that $U(A)^*=U(-A)=U(A)^{-1}$ so $U(A)$ is unitary and $U(\alpha A)U(\beta A)=U((a+\beta)A)$ for all $A\in\mathcal{A}$ and $\alpha,\beta\in\mathbb{R}$. Moreover if we let $u=U(0)$ we see that all elements of $\mathcal{P}$ can be expressed as $Hu$ where $H$ is a bounded operator on $\mathcal{F}$ belonging to the algebra generated by the $(U(A))_{A\in\mathcal{A}}$ and finally if we consider the state $\omega=|u\rangle\langle u|$ we have

$$\omega\left(\prod_k U(A_k)\right)=\left\langle u,\prod_k U(A_k)u\right\rangle=\exp\left(-\frac{1}{2}\sum_{1\leqslant j,k\leqslant n}Q(A_j,A_k)-i\sum_{1\leqslant j<k\leqslant n}\kappa(A_j,A_k)\right)$$

as we where looking for.

▷ The construction of an Hilbert space given an algebra and a state on it that we sketched here in our particular case can be generalised in the context of $C^*$ algebras and take the name of *Gelfand–Naimark–Segal construction*.

▷ Going back to our quantum Gaussian space $(\mathcal{F},\omega)$ we see that

$$\langle U(A+B),U(A)U(B)\rangle=\langle u,U(-(A+B))U(A)U(B)u\rangle=\exp(-i\kappa(A,B))$$

from which we deduce that

$$U(A+B)=\exp(i\kappa(A,B))U(A)U(B)$$

where this equality is understood in $\mathcal{H}$. Indeed

$$\|U(A+B)-\exp(i\kappa(A,B))U(A)U(B)\|^2=2-2\mathrm{Re}\langle U(A+B),\exp(i\kappa(A,B))U(A)U(B)\rangle=0.$$

Similarly we have

$$U(A)U(B)U(C)=U(A)\exp(-i\kappa(B,C))U(B+C)=\exp(-i\kappa(B,C)-i\kappa(A,B+C))U(A+B+C)$$

$$=\exp(-i\kappa(B,C)-i\kappa(A,B+C)+i\kappa(A+B,C))U(A+B)U(C)$$

$$=\exp(-i\kappa(A,B))U(A+B)U(C)$$

which implies that, as operators on $\mathcal{F}$:

$$U(A)U(B) = \exp(-i\kappa(A, B))U(A + B)$$

which implies also

$$U(A)U(B) = \exp(-2i\kappa(A, B))U(B)U(A).$$

This relation will play a very important role in the following and is called the Weyl form of the canonical commutation relations.

▷ From the Weyl relation we see that $\mathcal{P}$ is the span of all the operators in the form $U(A)$ and therefore that the state $\omega$ is determined by the relation

$$\omega(U(A)) = \exp\left(-\frac{1}{2}Q(A, A)\right).$$

▷ From the Weyl relations we deduce also that, if $\kappa = 0$, then the algebra generated by the $U(A_j)$ acting on $\mathcal{F}$ is commutative and therefore "corresponds" to a family of classical random variables with Gaussian law and covariance matrix $Q(A_i, A_j)$. To make this precise we would need to represent $U(A_j) = e^{i\varphi(A_j)}$ for some operator $\varphi(A_j)$. This will be unbounded (since it corresponds to a classical Gaussian). This problem will be dealt in the next section.