Sheet 6

(Lecture of 29/5/2018, 5/6 and 7/6/2018)

1 Quantum random variables

▷ **Three faces.** In this section we discuss three equivalent description of a quantum (real) observable:

1. A spectral measure $J: \mathcal{B}(\mathbb{R}) \to \mathcal{L}(\mathcal{H})$ on $\mathbb{R}$;
2. A strongly continuous unitary group $(U_t)_{t \in \mathbb{R}}$ acting on $\mathcal{H}$;
3. A self–adjoint operator $X: \mathcal{X} \subseteq \mathcal{H} \to \mathcal{H}$.

The set of all self–adjoint (not necessarily bounded) operators in an Hilbert space is not a nice space, for example sum of operators could not be self–adjoint. This difficulty is linked to the fact that unbounded self–adjoint operators are really spectral measures (or unitary groups) and, it is clear that spectral measures do not possess any natural vector space structure (if they do not commute).

Indeed we can think effectively to an unbounded operator X more as an homomorphism from the algebra $C(\mathbb{R})$ of bounded continuous functions on $\mathbb{R}$ to the bounded operators of $\mathcal{H}$ given by $X(f) = f(X)$ where the operator $f(X)$ is defined via the functional calculus associated to the spectral measure ξ of X. From this point of view is then clear that there is no natural vector space notion on these operators and that they are more complex and subtle than the more familiar bounded operators on $\mathcal{H}$. One could also think that calling them operators is somewhat of a misnomer.

Below we make precise the connections between the three manifestations of a quantum observable. We do not aim to give structured proof but more to trace a path to go from one description to the other. The reader can find precise statements and proofs in any book of functional analysis, for example in the first volume of Reed and Simon, or in the book of Parthasaraty.

▷ **From unitary groups to spectral measures and generators.** Let $(U_t)_{t \in \mathbb{R}}$ be a strongly continuous group of unitary operators on the Hilbert space $\mathcal{H}$. For any $f: \mathbb{R} \to \mathbb{R}$ with continuous and L^1 Fourier transform $\hat{f}$ we can defined the operator T_f by letting

$$T_f \psi = \int_{\mathbb{R}} \hat{f}(t)(U_t \psi) \mathrm{d}t$$

for all $\psi \in \mathcal{H}$. Norm continuity of $t \mapsto U_t \psi$ ensures the existence of the integral as an element in $\mathcal{H}$, moreover T_f is a bounded operator since

$$\|T_f \psi\| \leqslant \int_{\mathbb{R}} |\hat{f}(t)| \mathrm{d}t \|\psi\|,$$

it is self–adjoint

$$T_f^* = \int_{\mathbb{R}} \hat{f}(t) U_t^* \mathrm{d}t = \int_{\mathbb{R}} \hat{f}(-t) U_t \mathrm{d}t = \int_{\mathbb{R}} \hat{f}(t) U_t \mathrm{d}t = T_f,$$

and

$$T_f T_g = \int_{\mathbb{R}} \hat{f}(t) U_t \mathrm{d}t \int_{\mathbb{R}} \hat{g}(s) U_s \mathrm{d}s = \int_{\mathbb{R}} \hat{f}(t) \hat{g}(s) U_{t+s} \mathrm{d}t \mathrm{d}s = \int_{\mathbb{R}^2} \hat{f}(t-s) \hat{g}(s) U_t \mathrm{d}t \mathrm{d}s = T_{fg}.$$

For any unit vector $\psi \in \mathcal{H}$ the map $f \mapsto h_\psi(f) = \langle \psi, T_f \psi \rangle$ defines a positive linear functional on smooth compactly supported functions on $\mathbb{R}$. It is positive since

$$h_\psi(f^2) = \langle \psi, T_f T_f \psi \rangle = \| T_f \psi \| \geqslant 0.$$

Let $g_\varepsilon(t) = \exp(-\varepsilon t^2/2)$ then $\hat{g}_\varepsilon(t) = (2\pi\varepsilon)^{-1/2} \exp(-t^2/(2\varepsilon))$ and as $\varepsilon \to 0$,

$$h_\psi(g_\varepsilon) = \int_{\mathbb{R}} (2\pi\varepsilon)^{-1/2} \exp(-t^2/(2\varepsilon)) \langle \psi, U_t \psi \rangle \mathrm{d}t \to \|\psi\|^2 = 1.$$

So eventually $-g_\varepsilon \|f\|_\infty \leqslant f \leqslant g_\varepsilon \|f\|_\infty$ on all $\mathbb{R}$ and $-\|f\|_\infty h_\psi(g_\varepsilon) \leqslant h_\psi(f) \leqslant \|f\|_\infty h_\psi(g_\varepsilon)$ which implies taking $\varepsilon \to \infty$ that $|h_\psi(f)| \leqslant \|f\|_\infty$. Therefore the functional is continuous in the uniform topology and can be extended to all continuous functions vanishing at infinity. By Riesz representation theorem there exists a Borel probability measure μ_ψ on the one–point compactification $\bar{\mathbb{R}}$ of $\mathbb{R}$ such that

$$h_\psi(f) = \int_{\bar{\mathbb{R}}} f(x) \mu_\psi(\mathrm{d}x)$$

and since $\int_{\mathbb{R}} g_\varepsilon(x) \mu_\psi(\mathrm{d}x) \to 1$ as $\varepsilon \to 0$ this measure can be restricted as a probability measure on $\mathbb{R}$. From now on μ_ψ will denote such a restiction. If ψ is not normalized this measure has mass $\|\psi\|^2$. Observe that, for any pair $\psi, \varphi \in \mathcal{H}$,

$$\langle (\psi + \varphi), T_f(\psi + \varphi) \rangle - \langle \psi, T_f \psi \rangle - \langle \varphi, T_f \varphi \rangle = 2 \operatorname{Re} \langle \psi, T_f \varphi \rangle,$$

$$\langle (\psi + i\varphi), T_f(\psi + i\varphi) \rangle - \langle \psi, T_f \psi \rangle + \langle \varphi, T_f \varphi \rangle = 2 \operatorname{Im} \langle \psi, T_f \varphi \rangle,$$

and we can define the complex valued measure on $\mathbb{R}$,

$$\mu_{\psi,\varphi} = \frac{1}{2}(\mu_{\psi+\varphi} - \mu_\psi - \mu_\varphi) + \frac{i}{2}(\mu_{\psi+i\varphi} - \mu_\psi + \mu_\varphi),$$

such that T_f can be extended as a bounded operator to all bounded complex measurable functions by

$$\langle \psi, T_f \varphi \rangle = \int_{\mathbb{R}} f(x) \mu_{\psi,\varphi}(\mathrm{d}x),$$

with

$$\|T_f\|^2 = \sup_{\|\varphi\| \leqslant 1} \int_{\mathbb{R}} |f(x)|^2 \mu_\varphi(\mathrm{d}x) \leqslant \|f\|_\infty^2.$$

Moreover we can define a *spectral measure* J as the mapping $\mathcal{B}(\mathbb{R}) \to \mathcal{L}(\mathcal{H})$ given by

$$\langle \psi, J(A) \varphi \rangle = \int_A \mu_{\psi,\varphi}(\mathrm{d}x)$$

and observe that $J(A)$ is bounded, selfadoint and a projection (i.e. $J(A)^2 = J(A)$) and that moreover if $(A_k)_k$ is a increasing family of Borel sets then $J(\cup_k A_k) = \lim_{k \to \infty} J(A_k)$ in the strong operator tolopology since

$$\|J(A)\varphi - J(B)\varphi\| \leqslant \int_{\mathbb{R}} |\mathbb{I}_A - \mathbb{I}_B|^2 \mu_\varphi(\mathrm{d}x) = 2 \int_{\mathbb{R}} (1 - \mathbb{I}_{A \cap B}) \mu_\varphi(\mathrm{d}x) \to 0$$

by dominated convergence. Now observe that

$$\int_{\mathbb{R}} e^{isx} J(\mathrm{d}x) = \lim_{\varepsilon\to 0}\int_{\mathbb{R}} e^{isx} g_\varepsilon(x) J(\mathrm{d}x) = \lim_{\varepsilon\to 0}\int_{\mathbb{R}} \hat{g}_\varepsilon(t-s) U_t \mathrm{d}t = U_s.$$

Let Σ the closure of the set of points $x\in\mathbb{R}$ such that for some $\delta>0$ it holds $\sup_{\|\varphi\|\leqslant 1}\mu_\varphi(B(x,\delta))>0$ where $B(x,\delta)$ is the ball of radius δ around x. Then, if $B\cap\Sigma=\varnothing$ we have $J(B)=0$ and

$$T_f = \int_\Sigma f(x) J(\mathrm{d}x)$$

so that the operator T_f can be extended to functions f which are bounded on the set Σ.

Consider the set $\mathcal{X}\subseteq\mathcal{H}$ defined as

$$\psi\in\mathcal{X}\Leftrightarrow\int_{\mathbb{R}}|x|^2\mu_\psi(\mathrm{d}x)<+\infty.$$

This set is dense in $\mathcal{H}$, indeed $J([-t,t])\psi\to\psi$ as $t\to\infty$ and

$$\mu_{J([-t,t])\psi}(A) = \langle J([-t,t])\psi, J(A)J([-t,t])\psi\rangle = \langle\psi, J(A\cap[-t,t])\psi\rangle = \mu_\psi(A\cap[-t,t]).$$

On $\mathcal{X}$ we can define an operator X as

$$X\psi = \int_{\mathbb{R}} x\, J(\mathrm{d}x)\psi.$$

We have, for all $\psi\in\mathcal{X}$,

$$\lim_{t\to 0}\frac{U_t-1}{t}\psi = \lim_{t\to 0}\int_{\mathbb{R}}\frac{e^{itx}-1}{t}J(\mathrm{d}x)\psi \to \lim_{t\to 0}\int_{\mathbb{R}} ix\, J(\mathrm{d}x)\psi,$$

as a norm limit and by dominated convergence. Conversely if the limit exists and we call it η we have

$$\int_{[-L,L]} x^2\,\mu_\psi(\mathrm{d}x) = \left\|\int_{[-L,L]} x\, J(\mathrm{d}x)\psi\right\|^2 = \lim_{t\to 0}\left\|J([-L,L])\frac{U_t-1}{t}\psi\right\|^2 = \|J([-L,L])\eta\|^2 < +\infty,$$

so by monotone convergence $\psi\in\mathcal{X}$. So on $\mathcal{X}$ we can extend T_f to functions which grows linearly in x and we identify $T_x = X$ and $T_{e^{it\cdot}} = U_t = e^{itX}$.

Note also that for and for all $\psi\in\mathcal{H}$

$$\left\|\int_{\mathbb{R}}\frac{e^{itx}-1}{t}J(\mathrm{d}x)\psi\right\|^2 = \int_{\mathbb{R}}\left|\frac{e^{itx}-1}{t}\right|^2\mu_\psi(\mathrm{d}x) = \int_{\mathbb{R}}\frac{4\sin^2(tx/2)}{t^2}\mu_\psi(\mathrm{d}x).$$

Moreover for all $t>0$ we have

$$\left|\frac{4\sin^2(tx/2)}{t^2}\right| \leqslant |x|^2,$$

on one hand, and on the other hand, by Fatou lemma, for all $L>0$,

$$\int_{|x|\leqslant L} x^2\mu_\psi(\mathrm{d}x) = \int_{|x|\leqslant L}\liminf_{t\to 0}\frac{4\sin^2(tx/2)}{t^2}\mu_\psi(\mathrm{d}x)$$

$$\leqslant \liminf_{t\to 0}\int_{|x|\leqslant L}\frac{4\sin^2(tx/2)}{t^2}\mu_\psi(\mathrm{d}x) \leqslant \liminf_{t\to 0}\int_{\mathbb{R}}\frac{4\sin^2(tx/2)}{t^2}\mu_\psi(\mathrm{d}x) \leqslant \sup_t\left\|\frac{U_t-1}{t}\psi\right\|^2.$$

Therefore the domain $\mathcal{X}$ can be characterised also as the set of vectors for which $\|(U_t - 1)\psi\|^2/t^2$ is uniformly bounded as $t \to 0$.

The operator $X: \mathcal{X} \to \mathcal{H}$ is an unbounded, closed self–adjoint operator, i.e. $X = X^*$ including equality of domains. All the informations about it are carried by the spectral measure $J = J_X$ and by the family of unitarities $(U_t)_t = (U_t^X)_t$.

We call an observable the given of a spectral measure on $\mathcal{H}$ or equivalently of a one–parameter strongly continuous group of isometries, or a self–adjoint X. The set $\Sigma = \sigma(X) \subseteq \mathbb{R}$ is the *spectrum* of X and it is the support of the spectral measure J_X.

$\triangleright$ **and back.** Now let consider the situation where we start from the self–adjoint operator $(X, \mathcal{X})$ and we would like to reconstruct its spectral measure J_X and the unitary group $(e^{itX})_t$. In order to do so we first construct a bounded operator out of X.

It will be useful to consider first the more general situation where X is a symmetric closable operator defined on a dense set $\mathcal{X} \subseteq \mathcal{H}$. In this case its adjoint X^* with domain $\mathcal{X}^*$ is an extension of X.

Given an unbounded operator X we say that $z \in \mathbb{C}$ belongs to the resolvent set $\rho(X)$ of X if $(z - X)\varphi = f$ has a unique solution $\varphi \in \mathcal{X}$ for all $f \in \mathcal{H}$ and the *resolvent operator* $R(z) = R(z, X) = (z - X)^{-1}: f \mapsto \varphi$ is bounded.

Uniqueness is not a big issue here, indeed if $z = x + iy \in \rho(X)$ and $(z - X)\varphi = f$ then $\langle \varphi, f \rangle = \langle \varphi, (x - X)\varphi \rangle = \langle (x - X)\varphi, \varphi \rangle + iy\langle \varphi, \varphi \rangle$ and

$$|\langle \varphi, f \rangle|^2 = |\langle (x - X)\varphi, \varphi \rangle|^2 + y^2\|\varphi f\|^2.$$

Solutions to the equation $(z - X)\varphi = f$ are therefore unique if $y \neq 0$. Away of the real line the resolvent is well defined whenever exists and bounded by

$$\|R(z) f\| \leqslant |\mathrm{Im}\, z|^{-1}\|f\|.$$

For all $w, z \in \rho(X)$ we have

$$R(z) - R(w) = R(w)(w - z)R(z),$$

which implies that $[R(w), R(z)] = 0$. It also allows to constuct $R(z)$ from $R(w)$ if $|z - w|$ is small enough by perturbation theory for bounded operators in Banach spaces. Therefore $\rho(X)$ is an open set.

By perturbation theory we see also that if $w = x + iy$ and $z = x + iy'$ then whenever $|y - y'| < y$ we can construct $R(z)$ from $R(w)$ with the estimate

$$\|R(z)\| \leqslant (1 - |\mathrm{Im}\, w|^{-1}|w - z|)^{-1},$$

and similarly we can construct resolvent for $(x \pm \varepsilon y) + iy$ from the resolvent of $x + iy$. So whenever we have the resolvent in one point of the complex plane away from the real line we can extend it to all the point in the same half space (with positive or negative imaginary part).

Furthermore if $z \in \rho(X)$, for any $f \in \mathcal{H}$

$$\langle R(z) f, (\bar{z} - X)\varphi \rangle = \langle (z - X^*)R(z) f, \varphi \rangle = \langle (z - X)R(z) f, \varphi \rangle = \langle f, \varphi \rangle.$$

Therefore the equation $(\bar{z} - X)\varphi = g$ can have at most one solution for any $g \in \mathcal{H}$. Whenever it has one we have $\langle R(z) f, g \rangle = \langle f, \varphi \rangle$ so $\|\varphi\| \leqslant C\|g\|$ where C does not depends on g. If it has solution for any $g \in \mathcal{H}$ then $\bar{z} \in \rho(X)$ and then taking $\varphi = R(\bar{z})g$ we have $\langle R(z) f, g \rangle = \langle f, R(\bar{z})g \rangle$, that is $R(z)^* = R(\bar{z})$.

If $\mathrm{Ran}(z-X)\neq\mathcal{H}$ then there exists a vector $\psi\in\mathrm{Ran}(z-X)^{\perp}\cap\mathcal{X}$ such that for all $\varphi\in\mathcal{X}$,

$$0=\langle(z-X)\varphi,\psi\rangle=\langle\varphi,(\bar{z}-X^*)\psi\rangle$$

so $\psi\in\mathrm{Ker}(\bar{z}-X^*)$. Conversely if $\mathrm{Ker}(\bar{z}-X^*)\neq\{0\}$ then $\mathrm{Ran}(z-X)\neq\mathcal{H}$.

When X is self–adjoint we have $\mathrm{Ker}(\bar{z}-X^*)=\mathrm{Ker}(\bar{z}-X)=\{0\}$ since the equation $(\bar{z}-X)\varphi=0$ has at most one solution by the considerations above. Therefore $\mathrm{Ran}(z-X)=\mathcal{H}$ and the resolvent $R(z)$ is well defined and bounded and $\mathbb{C}\backslash\mathbb{R}\subseteq\rho(X)$. Boundedness comes from the closed graph theorem.

On the other hand if X is only symmetric but there exists $z\in\mathbb{C}$ such that $\mathrm{Ran}(z-X)=\mathcal{H}=\mathrm{Ran}(\bar{z}-X)$ (one can take $z=\pm i$) then it also holds that $\mathbb{C}\backslash\mathbb{R}\subseteq\rho(X)$.

In any case, when the resolvent is defined both in z and $\bar{z}$ we can continue to reason as follows. Take $z\in\rho(X)$ and denote with $\eta=\mathrm{Im}\,z$. We have

$$R(z)-R(\bar{z})=R(\bar{z})(\bar{z}-z)R(z)=2i\eta R(z)^*R(z),$$

and letting $U=1+2i\eta R(z)$ we have

$$U^*U=(1-2i\eta R(\bar{z}))(1+2i\eta R(z))=1+2i\eta(R(z)-R(\bar{z}))+4\eta^2R(\bar{z})R(z)$$

$$=1+2i\eta(R(z)-R(\bar{z})-2i\eta R(\bar{z})R(z))=1.$$

So the operator U is unitary. It is now not difficult to show that there exists a spectral measure θ on $\mathbb{D}=\{|z|=1\}\subseteq\mathbb{C}$ such that

$$U=\int_{\mathbb{D}}u\theta(\mathrm{d}u).$$

(How?)

Now we want to show that $\theta(\{1\})=0$. Assume that there exists $f\in\mathcal{H}$ such that $\theta(\{1\})f=f$. This implies

$$(1+2i\eta R(z))f=Uf=\int_{\mathbb{D}}u\theta(\mathrm{d}u)\theta(\{1\})f=f$$

so $R(z)f=0$ and then $f=0$. Consider now the function

$$g(u)=z-2i\eta\frac{1}{u-1},$$

such that

$$\overline{g(u)}=\bar{z}-2i\eta\frac{1}{\bar{u}-1}=z+2i\eta\Big[\frac{u}{u-1}-1\Big]=g(u)$$

and define the self–adjoint unbounded operator

$$Y=\int_{|u|=1,u\neq1}g(u)\theta(\mathrm{d}u).$$

We have also

$$R(z)=\frac{U-1}{2i\eta}=\int_{|u|=1,u\neq1}\frac{(u-1)}{2i\eta}\theta(\mathrm{d}u).$$

Now

$$YR(z)=\int_{|u|=1,u\neq1}g(u)\frac{(u-1)}{2i\eta}\theta(\mathrm{d}u)=\int_{|u|=1,u\neq1}(z-1)\theta(\mathrm{d}u)=z-1$$

So $R(z)$ is also the resolvent of Y in z. Moreover from the spectral decomposition is easy to see that $R(z)\mathcal{H}$ is dense in $\mathcal{H}$, that $R(z)\mathcal{H} \subseteq \mathcal{X} \cap \mathcal{Y}$ and finally that

$$0 = (z - Y)R_Y(z)g - g = (X - Y)R(z)g$$

so $X = Y$ on $R(z)\mathcal{H}$ and by density and closedness of the two operators we obtain $X = Y$ and deduce that X is self–adjoint. Finally, by a change of variable in the spectral integral we can construct a spectral measure J on $\mathbb{R}$ such that

$$X = \int_{\mathbb{R}} x\, J(\mathrm{d}\xi).$$

2 Structure of the quantum Gaussian

$\triangleright$ Up to now we paied attention to work only with the bounded unitary operators $U(A)$ while the quantum analog of the Gaussian should be the the operator $q(A)$ appearing in the expression $U(\alpha A) = \exp(i\alpha q(A))$. It's rigorous definition goes as follows. Note that

$$U(\alpha A)U(\beta A) = U((\alpha + \beta)A),$$

so for any A, the family $(U(\alpha A))_{\alpha\in\mathbb{R}}$ is a one–parameter group of unitary operators and, if we show that the family is strongly continuous, we can apply Stone's theorem and deduce that there exists on $\mathcal{F}$ a self–adjoint operator $q(A)$, the generator of the group $(U(\alpha A))_{\alpha\in\mathbb{R}}$, such that the formula $U(\alpha A) = \exp(i\alpha q(A))$ holds in the sense of functional calculus.

Strong continuity for unitary operators is equivalent to weak continutity (actually by a theorem of von Neumann weak measurability suffices, see Reed and Simon, Vol I, Theorem VIII.9), indeed if $(U_t)_t$ is a unitary group and f a vector

$$\|(U_t - U_s)f\|^2 = 2(\|f\|^2 - \mathrm{Re}\,\langle U_t f, U_s f\rangle) = 2(\|f\|^2 - \mathrm{Re}\langle f, U_{t-s} f\rangle) \to 0$$

whenever $\langle f, U_{t-s} f\rangle \to 0$.

Weak continuity for our family $(U(\alpha A))_{\alpha\in\mathbb{R}}$ on vectors in the span of the monomials in $(U(B))_B$ is easy to obtain from the from of the pseudo–characteristic function $\Phi(A)$ and then a density argument using that the operators $(U(\alpha A))_{\alpha\in\mathbb{R}}$ are unitary give easily weak continuity and then strong continuity.

Note that

$$\langle u, e^{i\alpha\varphi(A)} u\rangle = e^{-\alpha^2 Q(A,A)/2},$$

from which we can deduce that for any continuous and bounded function $f\colon \mathbb{R} \to \mathbb{R}$ we have

$$\langle u, f(\varphi(A))u\rangle = \int_{\mathbb{R}} f(x) e^{-x^2/(2Q(A,A))} \frac{\mathrm{d}x}{(2\pi Q(A,A))^{1/2}}.$$

Recall that the domain $\mathcal{D}(q(A))$ of $q(A)$ is the set of elements $\psi \in \mathcal{F}$ such that

$$\lim_{t\to 0} \Delta_t^A \psi = \lim_{t\to 0} \frac{U(tA) - 1}{t}\psi$$

exists in the norm sense or also such that $\|\Delta_t^A \psi\|$ is uniformly bounded. Now

$$\|\Delta_t^A u\|^2 = \frac{1}{t^2}(2 - 2\mathrm{Re}\langle U(tA)u, u\rangle) = \frac{1}{t^2}\left(2 - 2\exp\left(-\frac{1}{2}t^2 Q(A,A)\right)\right) \to Q(A,A) < +\infty,$$

so we have $u \in \mathcal{D}(q(A))$. Moreover

$$\Delta_t^A U(B)\psi = \frac{U(tA)-1}{t} U(B)\psi = U(B)\left(\frac{e^{-i\kappa(tA,B)}-1}{t} + \Delta_t^A\right)\psi$$

so

$$[\Delta_t^A, U(B)] = \frac{e^{-i\kappa(tA,B)}-1}{t} U(B)$$

and since $\frac{e^{-i\kappa(tA,B)}-1}{t} \to -i\kappa(tA,B)$ we see that $U(B)\mathcal{D}(q(A)) \subseteq \mathcal{D}(q(A))$ and

$$[q(A), U(B)] = \kappa(A,B)U(B).$$

Now

$$\langle u, \Delta_t^A u\rangle = \frac{1}{t}\langle u, (U(tA)u - u)\rangle = \frac{1}{t}\left[e^{-\frac{1}{2}Q(tA)} - 1\right] \to 0,$$

and

$$\langle U(B)u, \Delta_t^A u\rangle = \frac{1}{t}\langle u, U(-B)(U(tA)u - u)\rangle = \frac{1}{t}\left[e^{-i\kappa(-B,tA)}e^{-\frac{1}{2}Q(-B+tA)} - e^{-\frac{1}{2}Q(-B)}\right]$$

$$\to [i\kappa(B,A) + Q(B,A)]e^{-\frac{1}{2}Q(-B)} = \langle U(B)u, [i\kappa(B,A) + Q(B,A)]u\rangle,$$

so $q(A)u$ is given by the expression

$$\langle U(B)u, q(A)u\rangle = \langle U(B)u, [i\kappa(B,A) + Q(B,A)]u\rangle$$

Now, in general computations of expression of the form $\|\Delta_t^B\psi\|^2$ proceed in the same way and one can show that the domain $\mathcal{D}(q(A))$ contains all the vectors of the form

$$\left(\prod_j q(A_j)\right)u.$$

Otherwise stated, $u \in \mathcal{D}(\prod_j q(A_j))$ for all combinations of operators and that we have

$$\omega(q(A)q(B)) = Q(A,B) + i\kappa(A,B).$$

▷**Wick's theorem** More generally we have

$$\omega(q(A_1)\cdots q(A_{2n+1})) = 0$$

and

$$\omega(q(A_1)\cdots q(A_{2n})) = \sum_{(\alpha,\beta)} \prod_{i=1}^{n} \omega(q(A_{\alpha_i})q(A_{\beta_i}))$$

where the sum runs over all the pairings $((A_{\alpha_1}, A_{\beta_1}), \ldots, (A_{\alpha_n}, A_{\beta_n}))$ of the elements of the vector $(A_1, \ldots, A_{2n})$.

▷**A remark about the Quantum CLT.** The identification of the generators of the groups $(U(\alpha A))_\alpha$ is necessary to obtain the random variables object of our CLT. In particular now we can state results of the type weak convergence. For example we can say that for any two continuous functions $f, g: \mathbb{R} \to \mathbb{R}$ we have

$$\rho^{\otimes N}(f(\sigma_N(A))g(\sigma_N(B))) \to \langle u, f(\varphi(A))f(\varphi(B))u\rangle$$

as $N\to\infty$. Indeed we can approximate uniformly these functions with functions f_ε, g_ε such that $\hat{f}_\varepsilon(t)$, $\hat{g}_\varepsilon(t)$ are compactly supported and then observe that

$$\rho^{\otimes N}(f_\varepsilon(\sigma_N(A))g_\varepsilon(\sigma_N(B)))=\int_{\mathbb{R}^2}\hat{f}_\varepsilon(t)\hat{g}_\varepsilon(s)\rho^{\otimes N}(e^{it\sigma_N(A)}e^{is\sigma_N(B)})\mathrm{d}t\mathrm{d}s\to$$

$$\to\int_{\mathbb{R}^2}\hat{f}_\varepsilon(t)\hat{g}_\varepsilon(s)\langle u, e^{it\varphi(A)}e^{is\varphi(B)}u\rangle\mathrm{d}t\mathrm{d}s=\langle u, f_\varepsilon(\varphi(A))f_\varepsilon(\varphi(B))u\rangle$$

so taking away the approximation we deduce the claim.

3 Stone–von Neumann theorem

We now focus ourselves on a particular situation of two strongly continuous unitary groups $(U_t)_t$ and $(V_t)_t$ in an Hilbert space $\mathcal{H}$ satisfying the Weyl commutation relation

$$U_tV_s=e^{-ist}V_sU_t.$$

This is a particular case of our Quantum Gaussian space. We will prove now that this pair of operators is unitarily equivalent to a canonical model constructed on the Hilbert space $L^2(\mathbb{R})$.

Let J to be the spectral resolution associated to $(V_s)_s$ and with $T_f=f(X)$ where X is the generator of $(V_s)_s$ we have

$$U_tf(X)U_t^*=\int\hat{f}(s)U_tV_sU_t^*\mathrm{d}s=\int\hat{f}(s)e^{-ist}V_s\mathrm{d}s=f(X-t)$$

so in particular if we let $E_s=J((-\infty,s])$ we have $U_tE_sU_t^*=E_{s+t}$. Let $\mathcal{H}_t=E_t\mathcal{H}$ so that E_t is the orthogonal projection on $\mathcal{H}_t$.

▷**Screw lines.** We say that a curve $x\colon\mathbb{R}\to\mathcal{H}$ is a *screw line* in $\mathcal{H}$ if $x(0)=0$, $x(t)\in\mathcal{H}_t$ and $x(t)-x(s)\perp\mathcal{H}_s$ for all $t>s$ and $U_h(x(t)-x(s))=x(t+h)-x(s+h)$ for all $h>0$, $s<t$.

Theorem 1. *If $\mathcal{H}_{-\infty}\neq\mathcal{H}$ there exists a non–zero screw line.*

Proof. Assume that there are no nontrivial screw line. Let $\mathcal{E}=\mathcal{H}_0$ and $P_t=E_0U_t$ and note that it defined a strongly continuous semigroup of contractions of $\mathcal{E}$. The assumption $\mathcal{H}_{-\infty}\neq\mathcal{H}$ is equivalent to require that $\mathcal{H}_0\neq\mathcal{H}$. Let $(A,\mathcal{D}(A))$ the generator of P_t according to Hille–Yosida theory namely the closed, densely defined operator such that $\partial_{t+}P_t\psi=AP_t\psi$ for all $\psi\in\mathcal{D}(A)$. Take $f\in\mathcal{D}(A)\subseteq\mathcal{E}$ and observe that

$$x(t)=U_tf-U_0f-\int_0^tU_sAf\mathrm{d}s$$

is a screw line (better, can be extended as such). Indeed

$$E_sU_tf=E_sU_sU_{t-s}=U_sE_0U_{t-s}=U_sP_{t-s},$$

$$E_s(x(t)-x(s))=E_s(P_{t-s}f-P_0f)-\int_s^tE_sP_{r-s}Af\mathrm{d}r=0,$$

and

$$U_h(x(t)-x(s))=U_{t+h}f-U_{s+h}f-\int_s^tU_{r+h}Af\mathrm{d}s=x(t+h)-x(s+h).$$

Therefore we have $U_t f - U_0 f - \int_0^t U_s A f \mathrm{d}s = 0$ according to our assumption. Let X be the (Stone) generator of the group $(U_t)_t$ then for all $f \in \mathcal{D}(A)$ we have $f \in \mathcal{D}(X)$ and $i X f = A f$. Let

$$R_\lambda = \int_0^\infty e^{-\lambda t} P_t \mathrm{d}t, \qquad S_\lambda = \int_0^\infty e^{-\lambda t} U_t \mathrm{d}t.$$

For all $u \in \mathcal{E}$ we have $(\lambda + A) R_\lambda u = u$ therefore it also holds that $(\lambda + i X) R_\lambda u = u$ and as a consequence that $S_\lambda u = R_\lambda u$. From this we can deduce that $U_t = P_t$ on $\mathcal{E}$ and that U_t maps $\mathcal{E}$ in itself. But now if $f \in \mathcal{H}_s$ we also have $U_t f \in \mathcal{H}_{t+s}$ and therefore $\mathcal{H}_t \subseteq \mathcal{H}_0$ for all $t \geqslant 0$ which implies that $\mathcal{H} = \mathcal{H}_0$ contrary to assumptions. □

Remark 2. In order to claim that $P_t = U_t$ we use uniqueness of the Laplace transform. Note that for any two vectors $\psi, \varphi \in \mathcal{H}$ if we let $f(t) = \langle \psi, P_t \varphi \rangle - \langle \psi, U_t \varphi \rangle$ we have

$$\int_0^\infty e^{-\lambda t} f(t) \mathrm{d}t = 0, \tag{1}$$

for all $\lambda > 0$. The function f is continuous and bounded. If we conclude that $f(t) = 0$ we have proved that $P_t = U_t$ for all $t > 0$ since φ, ψ are arbitrary. Now we proceed by noting that (1) implies

$$\int_0^\infty e^{-\lambda_0 t} g(t) f(t) \mathrm{d}t = 0$$

for all g given by finite linear combinations of the form $g(t) = \sum c_m e^{-\lambda_m t}$ for arbitrary $(c_m \in \mathbb{R})_m$ and $(\lambda_m \geqslant 0)_m$. We denote by $\mathcal{L}$ the span of such functions. Here $\lambda_0 > 0$ is a fixed positive number. Fix a large $L > 0$ and note that $\mathcal{L}$ is also an algebra with unit which separates the points of $[0, L]$. Therefore by the Stone–Weierstrass theorem this algebra is dense in $C([0, L]; \mathbb{R})$ with the uniform norm. As a consequence we have that for any $g \in C_c([0, L]; \mathbb{R})$ eq. (1) also holds and since L is arbitrary it holds for all g compactly supported. At this point is enough to take a sequence of continuous functions $g_n \to \delta(t - t_0)$ weakly to deduce that $f(t_0) = 0$ for any $t_0 \in \mathbb{R}$.

▷ **The subspace generated by a screw line.** Now consider a non-zero screw line x. Note that by orthogonality of the increments we must have

$$\|x(t) - x(s)\|^2 = c|t - s|.$$

(why?) We can choose $c = 1$ without loss of generality. For any $f \in L^2(\mathbb{R})$ define the integral

$$I(f) = \int f(s) \mathrm{d}x(s),$$

in the same way we use to define the Ito integral, namely, observe that for simple f we have the isometry formula

$$\|I(f)\|^2 = \int |f(s)|^2 \mathrm{d}s,$$

and then extend the map by continuity. This is an isometric mapping of $L^2(\mathbb{R})$ into a closed subspace $\mathcal{M}_x$ of $\mathcal{H}$. This subspace is invariant under U_t and E_t and moreover

$$U_t I(f) U_t^* = U_t \int f(s) \mathrm{d}x(s) U_t^* = \int f(s) \mathrm{d}x(s + t) = \int f(s - t) \mathrm{d}x(s) = I(u_t f u_t^*)$$

and

$$E_t I(f) = E_t \int f(s) \mathrm{d}x(s) = \int \mathbb{I}_{s \leqslant t} f(s) \mathrm{d}x(s) = I(e_t f)$$

where we introduced two operators on $L^2(\mathbb{R})$ as $e_t f(s) = \mathbb{I}_{s \leqslant t} f(s)$ and $u_t f(s) = f(s-t)$.

Now if $\mathcal{M}_x^{\perp}$ is nontrivial we can extract another screw line and continue inductively until we exhaust the separable Hilbert space $\mathcal{H}$.

▷**The Schrödinger model.** At this point we realised that we can decompose $\mathcal{H}$ in a countable family of mutually orthogonal spaces isometric to $L^2(\mathbb{R})$ in such a way that U_t is sent to u_t and E_t to e_t. Seeing $(e_t)_t$ as a spectral repartition function we can also construct v_t as the image of V_t and observe that v_t acts as $v_t f(s) = e^{ist} f(s)$ on $L^2(\mathbb{R})$. Moreover note that the pair u, v satifies the Weyl relations as due

$$u_t v_s = e^{ist} v_s u_t.$$

This particular realization is called the Schrödinger model of the canonical commutation relations. Indeed calling p, q the generators of u, v respectively we have (at least formally for now)

$$[p, q] = i.$$

If U, V acts irreducibly then there can be only one factor in this direct sum decomposition.

▷**Irreducibility.** We want to prove now that the Schrödinger model is itself irreducible, that is the only stable subspaces for (E_t) and (U_t) are $\{0\}$ and $L^2(\mathbb{R})$. It will be sufficient to prove that there is ony one normalized screw line ξ in $L^2(\mathbb{R})$. Let $f = \xi \in L^2(\mathbb{R})$ since f is adapted we have $f(x) = 0$ if $x > 0$. Moreover since $(\xi(t) - \xi(0)) \perp e_0 L^2(\mathbb{R})$ we have $(\xi(t) - \xi(0))(x) = f(x+t) - f(x) = 0$ if $x < 0$. Therefore $f(x) = f(x+t)$ for all $t \geqslant 0$ and f is constant on $\mathbb{R}_-$. By normalization $f = \mathbb{I}_{[-\infty,0]}$ and $\xi(t) = \mathbb{I}_{[-\infty,t]}$ can be the only normalized screw line, proving irreducibility.

Therefore we have, in particular, proven the following theorem.

Theorem 3. (Stone–von Neumann) *Assume $(U_t)_t$ and $(V_t)_t$ are two strongly continuous unitary groups in an Hilbert space $\mathcal{H}$ satisfying the Weyl commutation relations*

$$U_t V_s = e^{-ist} V_s U_t, \qquad s, t \in \mathbb{R}$$

and acting irreducibly on $\mathcal{H}$ then there is an isomorphism of Hilbert spaces $\mathcal{U}\colon \mathcal{H} \to L^2(\mathbb{R})$ under which U_t, V_s are transformed into u_t, v_s.