Sheet 7

(Lecture of 12/6–14/6)

1 Quantum Gaussian (continued)

▷ **Necessity of infinite dimensional realization.** Conside a standard pair U_t, V_s We know that it is not possible to realise it on a finite dimensional Hilbert space $\mathcal{H}$, due to Stone–von Neumann theorem. Let us also see a different argument, which goes through Heisenberg's commutation relation. Let $U_t = e^{itP}$ and $V_s = e^{isQ}$ where P, Q are the self–adjoint Stone generators. If $\mathcal{H}$ is finite dimensional both operators are bounded and by differentiating twice the Weyl relations for U_t, V_s :

$$e^{itP} e^{isQ} = e^{ist} e^{isQ} e^{itP},$$

we deduce that P, Q should satisfy Heisenberg's commutation relation:

$$PQ - QP = [P, Q] = i.$$

Therefore we should also have for all $n \geqslant 1$

$$[P, Q^n] = PQ^n - Q^n P = PQ^n - QPQ^{n-1} + QPQ^{n-1} - Q^2 PQ^{n-2} + Q^2 PQ^{n-2} - \cdots - Q^n P$$

$$= [P, Q] Q^{n-1} + Q[P, Q] Q^{n-2} + \cdots + Q^{n-1} [P, Q] = inQ^{n-1}.$$

But now $\|[P, Q^n]\| \leqslant 2\|P\| \|Q^n\| \leqslant 2\|P\| \|Q\|^n$ and $\|Q^{n-1}\| = \|Q\|^{n-1}$. Moreover we know also that $\|Q\| \neq 0$ otherwise $[P, Q] = i$ cannot hold. So we conclude that $\|P\| \|Q\| \geqslant n$ for all $n \geqslant 0$, which is impossible if both operators are bounded. So one of the two has to be unbounded and $\mathcal{H}$ has to be infinite dimensional.

▷ **Irreducibility of the Schrödinger model.** Another way to prove irreducibility of the Schrödinger model is to observe that it is equivalent to the existence of two vectors $\psi, \varphi \in L^2(\mathbb{R})$ for which $0 = \langle \psi, U_t V_s \varphi \rangle$ for all $t, s \in \mathbb{R}$. Using the explicit nature of the Schrödinger model this matrix elements are given by

$$0 = \langle \psi, U_t V_s \varphi \rangle = \int_{\mathbb{R}} \bar{\psi}(x) e^{is(x-t)} \varphi(x-t) \mathrm{d}x = \int_{\mathbb{R}} \bar{\psi}(x+t) e^{isx} \varphi(x) \mathrm{d}x$$

Being this true for all s this implies the vanising of the Fourier transform of the L^1 function $h_t(x) = \bar{\psi}(x+t)\varphi(x)$ which in turn implies the vanishing of the function itself for almost all x and all $t \in \mathbb{R}$. Therefore $0 = \int_{\mathbb{R}} |h_t(x)|^2 \mathrm{d}x = \int_{\mathbb{R}} |\psi(x+t)|^2 |\varphi(x)|^2 \mathrm{d}x$ and integrating in t on $\mathbb{R}$ and using Fubini (which is allowed since everything is positive) we obtain

$$0 = \int_{\mathbb{R}} \mathrm{d}t \int_{\mathbb{R}} |h_t(x)|^2 \mathrm{d}x = \int_{\mathbb{R}} \left(\int_{\mathbb{R}} |\psi(x+t)|^2 \mathrm{d}t \right) |\varphi(x)|^2 \mathrm{d}x = \|\psi\|^2 \|\varphi\|^2$$

therefore $\psi = \varphi = 0$ and the model is irreducible.

▷ **Another proof of the Stone–von Neumann theorem.** There are various proof of this theorem, the following is taken from [Strocchi, F. *An Introduction to the Mathematical Structure of Quantum Mechanics: A Short Course for Mathematicians*. 2 edition. New Jersey: World Scientific Publishing Company, 2008.]

Introduce Weyl operators given by

$$W(s,t)=e^{-ist/2}V_tU_s=e^{ist/2}U_sV_t, \qquad s,t\in\mathbb{R},$$

for which $W(s,t)^*=(W(s,t))^{-1}=W(-s,-t)$ and

$$W(s,t)W(s',t')=e^{i(s't-st')/2}W(s+s',t+t')=e^{i(s't-st')}W(s',t')W(s,t).$$

Consider the operator

$$P:=\frac{1}{2\pi}\int_{\mathbb{R}^2}\mathrm{d}s\mathrm{d}t\, e^{-(t^2+s^2)/4}W(s,t)=P^*,$$

defined as strong limit of Riemman sums. We note that $P\neq 0$ Since otherwise

$$0=W(s',t')^*PW(s',t')=\frac{1}{2\pi}\int_{\mathbb{R}^2}\mathrm{d}s\mathrm{d}t\, e^{-(t^2+s^2)/4}e^{i(s't-st')}W(s,t)$$

which would imply that for any two vectors ψ, φ the function $(s,t)\mapsto e^{-(t^2+s^2)/4}\langle\psi, W(s,t)\varphi\rangle$ vanish and then in turn that $W(s,t)=0$. Next we note that

$$PW(s',t')P=e^{-((s')^2+(t')^2)/4}P$$

which can be proven by performing a change of variable in the Gaussian integrals in the l.h.s. This equations implies that $P^2=P$, i.e. P is a non-zero projection and therefore it exists a unit vector $\psi_0=P\psi_0$. If $\varphi\perp\psi_0$ and $P\varphi=\varphi$ then

$$\langle\psi_0,W(s,t)\varphi\rangle=e^{-(s^2+t^2)/4}\langle\psi_0,\varphi\rangle=0$$

and since the action is irreducible we deduce that $\varphi=0$. For the vector ψ_0 we have

$$\langle\psi_0,W(s,t)\psi_0\rangle=e^{-(s^2+t^2)/4}.$$

This expression is equivalent to the pseudo–caracteristic function for the generators of U_t and V_s. By using the GNS construction we deduce that there is an isomorphism of Hilbert spaces between any two irreducible realisations of the Weyl commutation relations. In particular they are all isomorphic to a given one (which can be taken to be the Schrödinger model).

$\triangleright$ **Reducible actions.** If $W(s,t)$ do not act irreducibly we do not have anymore that the range of P is one dimensional. However in general

$$\langle W(s',t')P\psi, W(s,t)P\varphi\rangle=\langle\psi, PW(-s',-t')W(s,t)P\varphi\rangle$$

$$=e^{i(s't-st')/2}\langle\psi, PW(s-s',t-t')P\varphi\rangle=e^{i(s't-st')/2}e^{-((s-s')^2+(t-t')^2)/4}\langle P\psi,P\varphi\rangle$$

$$=\langle W(s',t')\psi_0, W(s,t)\psi_0\rangle_{L^2}\langle P\psi,P\varphi\rangle$$

where there we used the irreducible canonical model to represent the numerical factor and where $\psi_0\in L^2(\mathbb{R})$ is a unit vector in the range of P for the canonical model. As a consequence the map $\mathcal{U}:\mathcal{E}\otimes\mathrm{Im}(P)\to\mathcal{H}$ given by

$$\mathcal{U}(W(s,t)\psi_0\otimes\varphi)=W(s,t)\varphi,$$

where $\mathcal{E}=\mathrm{span}(W(s,t)\psi_0: s,t\in\mathbb{R})\subseteq L^2(\mathbb{R})$ can be extended as an isometry of $L^2(\mathbb{R})\otimes\mathrm{Im}(P)\to\mathcal{H}$. Indeed $\mathcal{E}$ is dense in $L^2(\mathbb{R})$ due to the irreducibility of the Schrödinger model.

Moreover the image of $\mathcal{U}$ is all of $\mathcal{H}$, indeed if $\varphi\in\mathcal{H}$ is orthogonal to $\mathrm{Im}(\mathcal{U})$ we have for any $\psi\in\mathcal{H}$ and any $s',t'\in\mathbb{R}$

$$0=\langle\varphi,W(s',t')PW(-s',-t')\psi\rangle=\frac{1}{2\pi}\int_{\mathbb{R}^2}\mathrm{d}s\mathrm{d}t\,e^{-(t^2+s^2)/4}e^{i(s't-st')}\langle\varphi,W(s,t)\psi\rangle$$

therefore we have $\langle\varphi,W(s,t)\psi\rangle\,e^{-(t^2+s^2)/4}=0$ and as a consequence $\langle\varphi,W(s,t)\psi\rangle=0$ for all s,t since this function is continuous in s,t. But this implies $\langle\varphi,W(0,0)\psi\rangle=\langle\varphi,\psi\rangle=0$ and then $\varphi=0$ since this holds for any $\psi\in\mathcal{H}$.

So if the action of the Weyl pair is not irreducible we can state the Stone–von Neumann theorem as an isomorphism between $\mathcal{H}$ and $L^2(\mathbb{R})\otimes\mathcal{K}$ where the image of the Weyl pair acts canonically on the first factor and trivially on the second.

▷ **Plancherel.** Let U_t,V_s be the Schrödinger model on $L^2(\mathbb{R})$. Let $U_t'=V_{-t}$ and $V_s'=U_s$ then U', V' is another pair of strongly continuous unitary groups satisfying Weyl commutation relations and acting irreducibly on $L^2(\mathbb{R})$, therefore by the Stone–von Neumann theorem there exists an unitary operator on $L^2(\mathbb{R})$ which take this pair to the canonical pair U,V. This is the Fourier transform.

▷ **Other consequences.** Let $\tilde{W}(s,t)$ be a family satisfying the Weyl commutation relations on a separable Hilbert space $\mathcal{K}$. We have proven that each such pair is equivalent to a direct sum of copies of the Schrödinger model $W(s,t)$ acting on $L^2(\mathbb{R})$. In particular for every vector $f\in\mathcal{K}$ there exists vectors $f_n\in L^2(\mathbb{R})$ such that

$$\langle f,\tilde{W}(s,t)f\rangle_{\mathcal{K}}=\sum_n\langle f_n,W(s,t)f_n\rangle_{L^2(\mathbb{R})}=\mathrm{Tr}(\rho W(s,t))$$

which means that any law which can be constructed on a canonical pair (a pair satisfying Weyl relations) can be also contructed on the standard model. This constuction is equivalent to that of a canonical process in standard probability.