Sheet 8

(Lectures of 19/6–21/6)

▷ **Baker–Campbell formula.** Consider bounded operators X, Y. If $[X, Y]$ commutes with X, Y we have

$$e^X e^Y = e^{X+Y+\frac{1}{2}[X,Y]}.$$

This can be established as follows. Let $g(t) = e^{tX} e^{tY}$ and $h(t) = e^{tX} Y e^{-tX}$ then

$$g'(t) = X e^{tX} e^{tY} + e^{tX} Y e^{tY} = (X + e^{tX} Y e^{-tX}) g(t), \qquad h'(t) = e^{tX}[X, Y] e^{-tX} = [X, Y]$$

so $h(t) = Y + t[X, Y]$ and $g'(t) = (X + Y + t[X, Y]) g(t)$ which has solution

$$g(t) = e^{tX+tY+t^2/2[X,Y]}$$

from which we deduce the needed formula. Application to unbounded operators requires to deal with questions of domains and more specific hypothesis. We rely mainly on this formula for heuristic purposes in order to justify the construction of Weyl operators.

Note that

$$e^X e^Y = e^{X+Y+\frac{1}{2}[X,Y]} = e^Y e^X e^{[X,Y]}.$$

▷ **Reminder about Weyl operators.** Weyl operators are unitaries defined as follows. They corresponds at least formally to generators Q, P satisfying Heisenberg canonical commutation relations

$$[Q, P] = i.$$

$$W(s, t) = e^{-ist/2} e^{itQ} e^{isP} = e^{ist/2} e^{isP} e^{itQ} = e^{i(tQ+sP)}.$$

Now $[t'Q + s'P, tQ + sP] = i(t's - s't)$ so

$$W(s, t) W(s', t') = e^{i(t'Q+s'P)} e^{i(tQ+sP)} e^{-i(ts-s't)} = W(s', t') W(s, t) e^{-i(t's-s't)}.$$

If $z = s + it$ and we let $W(z) = W(s, t)$ then $\bar{z} z' = (s - it)(s' + it') = ss' + tt' + i(st' - s't)$ so

$$W(z) W(z') = W(z') W(z) e^{-i\mathrm{Im}(\bar{z} z')}.$$

1 Quantum Gaussian (continued)

▷ **General gaussian states.** Consider the tensor product of two copies of the Schrödinger model on which we define Weyl operators

$$\tilde{W}(s, t) f(x_1, x_2) = e^{-its/2} e^{-it(ax_1+bx_2)} f(x_1 - as, x_2 + bs)$$

where $a^2 - b^2 = 1$. Let now f_1, f_2 be two unit vector in $L^2(\mathbb{R})$ and consider their tensor product $f = f_1 \otimes f_2 \in L^2(\mathbb{R}^2)$. Then

$$\langle f_1 \otimes f_2, \tilde{W}(s,t) f_1 \otimes f_2 \rangle = \langle f_1, W(as, at) f_1 \rangle \langle f_2, W(-bs, bt) f_2 \rangle = e^{-(a^2+b^2)(s^2+t^2)/4}$$

where $a^2 + b^2 = 1 + 2b^2$. Therefore by choosing $b \geqslant 0$ we can construct Weyl pairs whose quantum caracteristic function is

$$e^{-\frac{1}{2}Q(s^2+t^2)}$$

for any $Q \geqslant 1/2$. Note that if $Q > 1/2$ the action is reducible while if $Q = 1/2$ it is irreducible. Indeed any irreducible action should have $Q = 1/2$ by Stone–von Neumann theorem.

It is not possible to have $Q < 1/2$. (why??)

Note that in the model described above there exist another Weyl pair, namely

$$W^\sharp(s,t) f(x_1, x_2) = e^{-its/2} e^{-it(bx_1 + ax_2)} f(x_1 + bs, x_2 - as)$$

and we have

$$\tilde{W}(s,t) W^\sharp(s',t') f(x_1,x_2) = e^{-its/2} e^{-it(ax_1+bx_2)} (W^\sharp(s',t') f)(x_1 - as, x_2 + bs)$$

$$= e^{-its/2} e^{-it(ax_1+bx_2)} e^{-it's'/2} e^{-it'(bx_1+ax_2)} e^{-it'(-bas+abs)} f(x_1 - as + bs', x_2 + bs - as')$$

$$= e^{-its/2} e^{-it(ax_1+bx_2)} e^{-it's'/2} e^{-it'(bx_1+ax_2)} f(x_1 - as + bs', x_2 + bs - as')$$

$$= W^\sharp(s',t') \tilde{W}(s,t) f(x_1,x_2)$$

so $W^\sharp$ and $\tilde{W}$ commute. Then we can construct the projection $P^\sharp$ associated to $W^\sharp$ and check that

$$P^\sharp \tilde{W} = \tilde{W} P^\sharp$$

and therefore $\tilde{W}$ cannot have an irreducible action as we already new.

Note also that if we denote by $\tilde{q}, \tilde{p}$ the generators of the Weyl pair $\tilde{W}$ and by q_1, p_1 and q_2, p_2 the generators of the two irreducible factors we have

$$\tilde{q} = a q_1 + b q_2, \quad \tilde{p} = a p_1 - b p_2,$$

at least formally and we note that $[\tilde{p}, \tilde{q}] = -i(a^2 - b^2) = -i$. The Weyl pair $W^\sharp$ corresponds to the linear transformation

$$q^\sharp = -b q_1 + a q_2, \quad \tilde{p} = -b p_1 - a p_2,$$

for which $(q^\sharp, p^\sharp)$ commute with $(\tilde{q}, \tilde{p})$.

▷ **Time–frequency analysis.** Let $f \in L^2(\mathbb{R})$. Observe that the quantum caracteristic function of the standard Weyl pair $W(s,t)$ on $L^2(\mathbb{R})$ with state f is given by

$$\langle f, W(s,t) f \rangle = \int_{\mathbb{R}} \bar{f}(x) e^{-ist/2} e^{itx} f(x-s) \mathrm{d}x = \int_{\mathbb{R}} \bar{f}(x + s/2) f(x - s/2) e^{itx} \mathrm{d}x.$$

In particular

$$e^{-(s^2+t^2)/4} = \langle \psi_0, W(s,t) \psi_0 \rangle = \int_{\mathbb{R}} \overline{\psi_0}(x + s/2) \psi_0(x - s/2) e^{itx} \mathrm{d}x.$$

so by Fourier transform we have

$$\overline{\psi_0}(x+s/2)\psi_0(x-s/2)e^{s^2/4} = \frac{e^{-x^2}}{(\pi)^{1/2}}$$

and letting $s=-2x$ we deduce $\overline{\psi_0}(0)\psi_0(2x) = \frac{e^{-2x^2}}{(\pi)^{1/2}}$ and therefore

$$\psi_0(x) = \frac{e^{-x^2/2}}{(\pi)^{1/4}}.$$

Now let $\gamma(\mathrm{d}x) = |\psi_0(x)|^2 \mathrm{d}x$ and transform $L^2(\mathrm{d}x)$ into $L^2(\gamma)$ by mapping $\psi \mapsto \psi/\psi_0$. This is a isomorphism between the two Hilbert spaces. Weyl operators become

$$\hat{U}_t f(x) = \psi_0^{-1}(x) U_t(f\psi_0)(x) = \psi_0^{-1}(x)(f\psi_0)(x-t) = e^{xt-t^2/2} f(x-t)$$

and

$$\hat{V}_s f(x) = \psi_0^{-1}(x) V_s(f\psi_0)(x) = e^{isx}\psi_0^{-1}(x)(f\psi_0)(x) = e^{isx} f(x).$$

This is the real Gaussian model of the Weyl pair.

2 Creation and annihilation operators

Let P, Q be the generators of the Weyl pair U, V on the Gaussian model $L^2(\gamma)$. It can be checked that

$$P = i(-\partial_x + x), \qquad Q = x,$$

$$[P, Q] = -i$$

on the span $\mathcal{P} \subseteq L^2(\gamma)$ of polynomials in the x variable which constitute a common dense set where the operators P, Q and their powers are well defined. The fact that it is dense follows easily from the Stone–Weierstrass theorem and an approximation argument.

Introduce the two operators $a^{\pm}$ by

$$Q = \frac{a^+ + a^-}{\sqrt{2}}, \qquad P = i\frac{a^+ - a^-}{\sqrt{2}}.$$

Note that

$$a^+ = \frac{Q - iP}{\sqrt{2}} = \frac{1}{\sqrt{2}}(2x - \partial_x), \quad a^- = \frac{Q + iP}{\sqrt{2}} = \frac{1}{\sqrt{2}}\partial_x$$

and

$$a^+a^- - a^-a^+ = [a^+, a^-] = -1.$$

The operators a^+, a^- are densely defined and adjoint to each other, therefore they are closable. Define also

$$a^\circ = a^+a^- = \frac{1}{2}(2x\partial_x - \partial_x^2).$$

Let h_0 be the vector such that $a^- h_0 = 0$, it is unique and $h_0(x) = 1$. And let $h_n = (a^+)^n h_0$, they are eigenvectors for a° and

$$a^\circ h_n = a^+a^-(a^+)^n h_0 = n h_n.$$

Moreover

$$\langle h_n, h_n\rangle = \langle a^+ h_{n-1}, a^+ h_{n-1}\rangle = \langle h_{n-1}, a^- a^+ h_{n-1}\rangle = \langle h_{n-1}, (a^+ a^- + 1) h_{n-1}\rangle = n\langle h_{n-1}, h_{n-1}\rangle$$

and $\langle h_n, h_n\rangle = n!$. The functions $(h_n)_n$ are polynomials and h_n has maximum degree n which implies that in their span there are all the polynomials and as a consequence they are dense in $L^2(\mathbb{R})$. The set $(h_n/(n!)^{1/2})_n$ is an orthonormal basis of $L^2(\gamma)$ which diagonalises the operator a° which is therefore self-adjoint (since it has a spectral decomposition supported on $\mathbb{R}$).

Next if we let $z = s + it$, let $W(z) = W(s,t)$ and consider the unit vectors

$$(W(z)h_0)(x) = e^{-ist/2} e^{itx}\, e^{xs - s^2/2} = e^{-ist/2 - s^2/2} e^{xz} = e^{xz} e^{-z^2/4 - |z|^2/4}$$

since $z^2 + |z|^2 = (s+it)^2 + s^2 + t^2 = 2s^2 + 2ist$.

On the other hand we have also by BCH formula

$$e^{-ist/2} e^{itQ} e^{-isP} = e^{ist/2} e^{itQ - isP} e^{-\frac{1}{2}[Q,P]st} = e^{itQ - isP} = e^{it(a^+ + a^-) + s(a^+ - a^-)} = e^{za^+ - \bar{z}a^-}$$

$$W(z) = e^{za^+ - \bar{z}a^-} = e^{-|z|^2/2} e^{za^+} e^{-\bar{z}a^-},$$

and then

$$W(z)h_0 = e^{-|z|^2/2} e^{za^+} e^{-\bar{z}a^-} h_0 = e^{-|z|^2/2} e^{za^+} h_0 = e^{-|z|^2/2} \sum_n \frac{z^n}{n!} (a^+)^n h_0 = e^{-|z|^2/2} \sum_n \frac{z^n}{n!} h_n$$

so we conclude that

$$e^{-|z|^2/2} \sum_n \frac{z^n}{n!} h_n(x) = e^{-|z|^2/4} e^{xz - z^2/4} = \cdots$$

from which we can obtain expressions for the h_n. (there is a problem with the exponents...)

We introduce also *exponential vectors*

$$\mathcal{E}(z)(x) = (e^{za^+} h_0)(x) = \sum_n \frac{z^n}{n!} h_n(x) = e^{xz - z^2/4}$$

for which

$$\mathcal{E}(0) = 1, \qquad \langle \mathcal{E}(z), \mathcal{E}(w)\rangle = \sum_n \frac{(\bar{z}\,w)^n}{n!} = e^{\bar{z}w}.$$

The linear space $\mathcal{E}$ generated by all the exponential vectors is dense in $\mathcal{H}$ since it contains the multiple derivatives of $\mathcal{E}(z)$ wrt. z and therefore the vectors $(h_n)_n$ which are themselves dense.

Weyl operators have an explicit simple action on exponential vectors

$$W(z)\mathcal{E}(u) = e^{-\bar{z}u - |z|^2/2} \mathcal{E}(u + z)$$

which can be interpreted also in an infinite dimensional context without difficulties. Actually we can introduce a family of extended Weyl operators as follows

$$W(z,\lambda)\mathcal{E}(u) = e^{-e^{i\lambda}\bar{z}u - |z|^2/2} \mathcal{E}(u + e^{i\lambda} z)$$

which encode a further action of $\mathbb{R}$ with unitary operators on $\mathbb{C}$.

▷ **Complex gaussian model.** For every $h \in \mathcal{H}$ we can consider the function $\varphi_h(z) = \langle h, \mathcal{E}(z)\rangle$ which is an antilinear, injective mapping from $\mathcal{H}$ to the space of entire functions. Exponential vector $\mathcal{E}(u)$ is mapped to the function $e^{\bar{u}z}$, a^+ to the multiplication with the function z and a^- to the derivation ∂_z. Scalar product is given by

$$\langle z^n, z^m\rangle = \delta_{n,m} n!$$

which coincides with the scalar product

$$\langle z^n, z^m\rangle = \int_{\mathbb{C}} (z^n)^* z^m e^{-|z|^2} \frac{\mathrm{d}z}{\pi}.$$

Therefore $\mathcal{H}$ is equivalent to the L^2 space of entire function with this scalar product. This is the Bargmann–Segal space, complex wave representation. Weyl operators are represented as

$$\varphi_{W(u)h}(z) = \langle h, W(-u)\mathcal{E}(z)\rangle = e^{\bar{u}z - |u|^2/2}\langle h, \mathcal{E}(z-u)\rangle = e^{\bar{u}z - |u|^2/2}\varphi_h(z-u)$$

very similar to the action in the real Gaussian representation.

▷ **Non–vacuum states.** Consider now states generated by exponential vectors. We have

$$e^{-|w|^2}\langle \mathcal{E}(w), W(z)\mathcal{E}(w)\rangle = e^{-|w|^2}\langle \mathcal{E}(w), e^{-\bar{z}w - |z|^2/2}\mathcal{E}(w+z)\rangle = e^{\bar{w}z - \bar{z}w - |z|^2/2}$$

Recalling that $W(s+it) = e^{itQ - isP}$ we have

$$e^{-|w|^2}\langle \mathcal{E}(w), e^{itQ}\mathcal{E}(w)\rangle = e^{2i\mathrm{Re}(w)t - t^2/2}$$

and similarly for P. So in these new states P, Q are Gaussian random variables with non-zero mean. Let us compute the distribution of the number operator $a^\circ = a^+ a^-$.

$$e^{-|w|^2}\langle \mathcal{E}(w), e^{-ita^\circ}\mathcal{E}(w)\rangle = e^{-|w|^2}\sum_n \frac{|w|^2}{n!}\langle h_n, e^{-ita^\circ}h_n\rangle = e^{-|w|^2}\sum_n \frac{|w|^2}{n!}e^{-itn} = e^{|w|^2(e^{-it}-1)}$$

which is a Poisson law of parameter $|w|^2$. Classical states are obtained by taking density matrices obtaines as

$$\rho = \int_{\mathbb{C}} |\mathcal{E}(z)\rangle\langle\mathcal{E}(z)| \mu(\mathrm{d}z)$$

for some measure μ on $\mathbb{C}$. Taking a Gaussian measure one obtains all the Gaussian states for the canonical pair. In particular if we take $\lambda > 0$ and

$$\mu(\mathrm{d}z) = C\frac{\lambda}{\pi}e^{-\lambda|z|^2}\mathrm{d}z$$

we have

$$\rho = C\int_{\mathbb{C}} |\mathcal{E}(z)\rangle\langle\mathcal{E}(z)| \frac{\lambda}{\pi}e^{-\lambda|z|^2}\mathrm{d}z = C\sum_n \frac{|h_n\rangle\langle h_n|}{(n!)^2}\int_{\mathbb{C}} |z|^{2n}\frac{\lambda}{\pi}e^{-\lambda|z|^2}\mathrm{d}z = C\sum_n \frac{|h_n\rangle\langle h_n|}{n!}\lambda^{-2n} = C\lambda^{-2a^\circ}$$

so letting $\lambda = e^{t/2}$ we have $\rho = Ce^{-ta^\circ}$ for $t > 0$ where C is fixed by requiring $\mathrm{Tr}(\rho) = 1$. But these computations can be justified only if $\lambda > 1$ indeed we have

$$\|\langle h, \mathcal{E}(z)\rangle\| \leqslant \|h\|\,\|\mathcal{E}(z)\| \leqslant \|h\| e^{|z|^2/2}$$

so

$$|\langle f,\rho g\rangle|=\left|\int_{\mathbb{C}}\langle f,\mathcal{E}(z)\rangle\langle\mathcal{E}(z),g\rangle\frac{\lambda}{\pi}e^{-\lambda|z|^2}\mathrm{d}z\right|\leqslant\|f\|\,\|g\|\int_{\mathbb{C}}\frac{\lambda}{\pi}e^{(1-\lambda)|z|^2}\mathrm{d}z\lesssim\|f\|\,\|g\|$$

so the density matrix is a bounded operator if $\lambda>1$. Moreover it has finite trace. We have deduced the formula

$$e^{-ta^{\circ}}=\int_{\mathbb{C}}|\mathcal{E}(z)\rangle\langle\mathcal{E}(z)|\frac{e^{t/2}}{\pi}e^{-e^{t/2}|z|^2}\mathrm{d}z.$$

When $t\to 0$ this formula formally becomes

$$\mathrm{Id}=\int_{\mathbb{C}}|\mathcal{E}(z)\rangle\langle\mathcal{E}(z)|\frac{1}{\pi}e^{-|z|^2}\mathrm{d}z$$

which expresses the fact that $(\mathcal{E}(z)e^{-|z|^2/2})_{z\in\mathbb{C}}$ is an (over)complete basis for $\mathcal{H}$. In particular we have the formula

$$h=\lim_{t\to 0}e^{-ta^{\circ}}h=\lim_{t\to 0}\int_{\mathbb{C}}\mathcal{E}(w)\overline{\varphi_h(w)}\frac{e^{t/2}}{\pi}e^{-e^{t/2}|w|^2}\mathrm{d}w$$

where the limit is a norm limit and the integral is understood as a Bochner integral, indeed note that

$$|\varphi_h(w)|\leqslant\|h\|e^{|w|^2/2}$$

pointwise in $w\in\mathbb{C}$ and that we have also $\|\mathcal{E}(w)\|\leqslant e^{|w|^2/2}$.

Note that $\varphi_h(w)$ is an analytic function (in fact, entire) so if it vanish on a curve of non-zero lenght it should vanish everywhere. This fact is linked to the overcompleteness of this basis. Related set which ensure vanishing of the function are the lattices $\gamma(\mathbb{Z}+i\mathbb{Z})$ for any $\gamma\in(0,\sqrt{\pi}]$. We will not show this. The case $\gamma=\sqrt{\pi}$ is called von Neumann lattice. When $\gamma>\sqrt{\pi}$ the lattice is not dense enough to guarantee this property.

If B is a bounded operator, we have

$$\langle\mathcal{E}(z),B\mathcal{E}(w)\rangle=\sum_{n,m}\frac{\bar{z}^n w^m}{n!m!}\langle h_n,Bh_m\rangle=F_B(\bar{z},w)$$

and we can recover the values of the coefficients $\langle h_n,Bh_m\rangle$ from the knowledge of the function $F_B(\bar{z},z)$ (How?). Therefore the operator B is completely determined by $F_B(\bar{z},z)=\langle\mathcal{E}(z),B\mathcal{E}(z)\rangle$, namely by its diagonal. Also, for trace class operators,

$$\mathrm{Tr}(B)=\int_{\mathbb{C}}\langle\mathcal{E}(z),B\mathcal{E}(z)\rangle e^{-|z|^2}\frac{\mathrm{d}z}{\pi}.$$

▷ **Action of a°.** We want to show now that

$$e^{ita^{\circ}}W(z)e^{-ita^{\circ}}=W(e^{it}z)$$

and

$$W(-z)e^{ita^{\circ}}W(z)=e^{it(a^{\circ}+za^{+}+\bar{z}a^{-}+|z|^2)}.$$

Recall the action of W on the exponential vectors

$$W(z)\mathcal{E}(u)=e^{-\bar{z}u-|z|^2/2}\mathcal{E}(u+z)$$

and the action of e^{ita° which can be computed simply by

$$e^{ita^\circ}\mathcal{E}(u) = e^{ita^\circ}\sum_n \frac{u^n}{n!}h_n = \sum_n e^{itn}\frac{u^n}{n!}h_n = \mathcal{E}(e^{it}u).$$

Then

$$e^{ita^\circ}W(z)e^{-ita^\circ}\mathcal{E}(u) = e^{ita^\circ}W(z)\mathcal{E}(e^{-it}u) = e^{ita^\circ}e^{-\bar{z}e^{-it}u-|z|^2/2}\mathcal{E}(e^{-it}u+z)$$

$$= e^{-\bar{z}e^{-it}u-|z|^2/2}\mathcal{E}(u+e^{it}z) = W(e^{it}z)\mathcal{E}(u).$$

Next, the family $(Q_t)_t := (W(-z)e^{ita^\circ}W(z))_t$ is a one parameter group of unitary transformations, let us compute its generator. We have

$$W(-z)e^{ita^\circ}W(z)\mathcal{E}(u) = W(-z)e^{ita^\circ}e^{-\bar{z}u-|z|^2/2}\mathcal{E}(u+z) = W(-z)e^{-\bar{z}u-|z|^2/2}\mathcal{E}(e^{it}u+e^{it}z)$$

$$= e^{-\bar{z}u-|z|^2/2}e^{\bar{z}(e^{it}u+e^{it}z)-|z|^2/2}\mathcal{E}(e^{it}u+e^{it}z-z) = e^{\bar{z}u(e^{it}-1)+(e^{it}-1)|z|^2}\mathcal{E}(e^{it}u+(e^{it}-1)z)$$

Therefore

$$\frac{\mathrm{d}}{\mathrm{d}t}|_{t=0}W(-z)e^{ita^\circ}W(z)\mathcal{E}(u) = \frac{\mathrm{d}}{\mathrm{d}t}|_{t=0}e^{\bar{z}u(e^{it}-1)+(e^{it}-1)|z|^2}\sum_{n\geqslant 0}\frac{(e^{it}u+(e^{it}-1)z)^n}{n!}h_n$$

$$= i(u+z)\sum_{n\geqslant 1}\frac{u^{n-1}}{(n-1)!}h_n + i(\bar{z}u+|z|^2)\sum_{n\geqslant 0}\frac{u^n}{n!}h_n$$

$$= ia^\circ\sum_{n\geqslant 1}\frac{u^n}{n!}h_n + iza^+\sum_{n\geqslant 0}\frac{u^{n-1}}{(n-1)!}h_{n-1} + i\bar{z}a^-\sum_{n\geqslant 0}\frac{u^{n+1}}{(n+1)!}h_{n+1} + i|z|^2\sum_{n\geqslant 0}\frac{u^n}{n!}h_n$$

$$= i(a^\circ + za^+ + \bar{z}a^- + |z|^2)\mathcal{E}(u)$$

as claimed. One can use von Neumann's theorem on analytic vectors (below) to check that $Z = a^\circ + za^+ + \bar{z}a^- + |z|^2$ is essentially selfadjoint on the span of $(h_n)_n$ and therefore its closure coincides with the generator of $(Q_t)_t$. The law of Z in the vacuum state is that of a Poisson random variable with intensity $|z|^2$.

Theorem 1. (Nelson) *Assume that the symmetric operator A has a dense subspace $\Phi \subseteq \mathcal{D}(A)$ of analytic vectors then it is essentially selfadjoint. Here a vector $\phi \in \mathcal{D}(A)$ is analytic for A iff $A^n\phi \in \mathcal{D}(A)$ for all $n \geqslant 0$ and $\sum_{n\geqslant 0}\|A^n\phi\|\frac{t^n}{n!} < \infty$ for some $t > 0$.*