Sheet 9

(Lectures of 2/7–4/7)

So far we discussed ncCLT and ncGaussians in one dimension. The theory of Weyl operators in finite dimensions is very similar and is left to the student. In particular Stone–von Neumann theorem holds and all the irreducible realisations of the Weyl operators are unitarily equivalent. In order to discuss the equivalent of stochastic processes in continuous time, and in particular to introduce the non-commutative analog of the Brownian motion we would need infinite dimensional Weyl operators.

▷ **Weyl operators indexed by Hilbert space.** Weyl operators $(W(h))_h$ can be meaningfully defined with h ranging over a complex Hilbert space $\mathcal{H}$ and requiring that

$$W(h)W(h') = e^{-i\mathrm{Im}\langle h,h'\rangle/2}W(h+h').$$

Whenever we consider only finite dimensional subspaces of $\mathcal{H}$, the corresponding Weyl operators can be realised, according to Stone–von Neumann theorem, only in essentially one way, as we have seen in the previous lectures. As we move to infinite dimensions however this is not anymore true. There are irreducible representations which are not unitarily equivalent. This has to be compared to the fact that it is "easy" for Gaussian measures in infinite dimensions to be mutually singular.

Inspired by our ncCLT we could consider the vector space $\mathcal{W}$ generated by elements $(W(h))_{h\in\mathcal{H}}$ with a scalar product fixed by letting

$$\langle W(h), W(h')\rangle = e^{i\mathrm{Im}\langle h,h'\rangle/2}e^{-\frac{1}{4}\|h'-h\|^2}.$$

Note that $\frac{1}{2}\|h'-h\|^2 - i\mathrm{Im}\langle h,h'\rangle = \frac{1}{2}\|h\|^2 + \frac{1}{2}\|h'\|^2 + \langle h',h\rangle$.

This scalar product is positive definite : one can see this using the ncCLT we proved or otherwise by noting the following.

Consider the Hilbert space $(\mathbb{C}\oplus\mathcal{H})^n$ and let $\varphi_n(h) = \left(1\oplus\frac{1}{2^{1/2}n^{1/2}}h\right)^{\otimes n}$ then

$$\langle\varphi_n(h'),\varphi_n(h)\rangle = \sum_{k=0}^{n}\frac{1}{n^k}\frac{n!}{(n-k)!k!}\langle h',h\rangle^k = \sum_{k=0}^{n}\frac{n(n-1)\cdots(n-k+1)}{n^k}\frac{\langle h',h\rangle^k}{2^k k!} \to e^{\langle h',h\rangle/2}$$

which shows that the function $e^{\langle h',h\rangle/2}$ is positive definite. Therefore we can introduce the scalar product

$$\left\langle\sum_i \lambda_i W(h_i), \sum_j \lambda_j W(h_j')\right\rangle = \sum_{i,j}\bar{\lambda}_i\lambda_j\, e^{-\frac{1}{4}\|h_i\|^2}e^{-\frac{1}{4}\|h_j'\|^2}e^{\langle h_j',h_i\rangle/2}$$

and complete $\mathcal{W}$ wrt. this scalar product to obtain an Hilbert space $\mathcal{F}$. We denote by Φ the vector corresponding to $W(0)$ and by abuse of notation, $W(h)$ the unitary Weyl operators.

We let also $\mathcal{E}(h) = e^{\frac{1}{4}\|h\|^2}W(h)\Phi$ the exponential vectors, for which

$$\langle\mathcal{E}(h),\mathcal{E}(h')\rangle = e^{\langle h,h'\rangle/2}.$$

Therefore we have a realization of the Weyl algebra on the quantum probability space $(\mathcal{F}, \Phi)$ for which

$$W(h)\mathcal{E}(h') = e^{-\langle h, h'\rangle/2 - \frac{1}{4}\|h\|^2}\mathcal{E}(h + h'), \qquad \langle \Phi, W(h)\Phi\rangle = e^{-\frac{1}{4}\|h\|^2}.$$

The vectors $(\mathcal{E}(h))_h$ are linearly independent and span a dense set in $\mathcal{F}$. Density is clear by construction. Linear independence can be seen as follows. Assume that $\sum_i \lambda_i \mathcal{E}(h_i) = 0$ for some choice of $(\lambda_i)_i \subseteq \mathbb{C}$ and $(h_i)_i \subseteq \mathcal{H}$. Then for all $g \in \mathcal{H}$ and $t \in \mathbb{R}$ we have

$$0 = \langle \mathcal{E}(tg), \sum_i \lambda_i \mathcal{E}(h_i)\rangle = \sum_i \lambda_i e^{t\langle g, h_i\rangle/2}$$

by chooising g appropriately we can assume that there exists a unique j such that $\langle g, h_j\rangle > 0$ and $\langle g, h_j\rangle > |\langle g, h_i\rangle|$ for all $i \neq j$. Therefore by sending $t \to \infty$ we deduce that

$$|\lambda_j| \leqslant \sum_{i \neq j} |\lambda_i| e^{t(|\langle g, h_i\rangle| - \langle g, h_j\rangle)/2} \to 0.$$

By continuing in this way with the remaining non-zero coefficients, we conclude that all the coefficients must be zero.

▷ For each unit vector $h \in \mathcal{H}$ we can consider the Weyl pair $U_s = W(sh)$, $V_t = W(ith)$. Indeed we have

$$U_s V_t = e^{-ist} V_t U_s, \qquad \langle \Phi, U_s \Phi\rangle = \langle \Phi, V_s \Phi\rangle = e^{-\frac{s^2}{4}}.$$

In turn to (U_s, V_t) it corresponds self–adjoint operators P_h, Q_h satisfying canonical commutation relations and creation and annihilation operators $a_h^{\pm}$.

More generally we can establish that $P_h = Q_{ih}$ and that $Q_{k+l} = Q_k + Q_l$ if $\mathrm{Im}\,\langle k, l\rangle = 0$ and

$$[Q_h, Q_k] = i\,\mathrm{Im}\langle h, k\rangle.$$

Note that

$$[a_h^+, a_k^+] = 0, \qquad [a_h^-, a_k^-] = 0, \qquad [a_h^-, a_k^+] = \langle h, k\rangle.$$

▷ **Fock space.** The vacuum vector Φ is annihilated by all the $(a_h^-)_h$ and we have $\mathcal{E}(h) = e^{a_h^+}\Phi$. In particular the family of vectors $((a_h^+)^n \Phi)_{n,h}$ span a dense set in $\mathcal{F}$, by polarization the same is true for the vectors

$$a_{h_1}^+ \cdots a_{h_n}^+ \Phi, \qquad h_1, \ldots, h_n \in \mathcal{H}$$

which are ortonogal for different n and have norms

$$\|a_{h_1}^+ \cdots a_{h_n}^+ \Phi\|^2 = \sum_\sigma \prod_{i=1}^n \langle h_i, h_{\sigma(i)}\rangle = \frac{1}{n!} \sum_{\sigma, \sigma'} \prod_{i=1}^n \langle h_{\sigma'(i)}, h_{\sigma(i)}\rangle$$

where σ, σ' runs over the permutations of $\{1, \ldots, n\}$. From this formula we realise that the Hilbert space $\mathcal{F}$ is isomorphic to the direct sum

$$\mathcal{F} \simeq \oplus_{n=0}^\infty \mathcal{H}^{\otimes_s n}$$

where $\mathcal{H}^{\otimes_s n}$ denotes the n-th symmetric tensor product of $\mathcal{H}$ which can be realised by taking the full tensor product and projecting on the symmetric component. Indeed observe that

$$\|a_{h_1}^+ \cdots a_{h_n}^+ \Phi\|^2 = n! \langle \Pi_s(\otimes_i h_i), \Pi_s(\otimes_i h_i)\rangle_{\mathcal{H}^{\otimes n}}$$

where $\Pi_s(\otimes_i h_i) = \frac{1}{n!}\sum_\sigma \otimes_i h_{\sigma(i)}$ is the symmetrization operator. Note also the factor $n!$ which remains and which relates the norm on $\mathcal{F}$ to the natural norm of $\oplus_{n=0}^\infty \mathcal{H}^{\otimes n}$ before symmetrization.

From now on we will realize concretely $\mathcal{F}$ as $\oplus_{n=0}^\infty \mathcal{H}^{\otimes_s n}$ with this scalar product. The vacuum vector corresponds to a unique (modulo phase factor) unit vector in the factor with $n=0$, so we will fix $\Phi \in \mathcal{H}^{\otimes_s 0} \subseteq \mathcal{F}$. With this identification we can now write, for example, the exponential vectors as

$$\mathcal{E}(h) = \sum_{n=0}^\infty \frac{h^{\otimes n}}{n!} \in \oplus_{n=0}^\infty \mathcal{H}^{\otimes_s n} = \mathcal{F}.$$

We denote also with $h_1 \circ \cdots \circ h_n = \Pi_s(h_1 \otimes \cdots \otimes h_n)$ the symmetric tensors for which

$$\|h_1 \circ \cdots \circ h_n\|_{\mathcal{F}}^2 = n! \|\Pi_s(h_1 \otimes \cdots \otimes h_n)\|_{\mathcal{H}^{\otimes n}}^2 = \sum_\sigma \prod_{i=1}^n \langle h_i, h_{\sigma(i)}\rangle.$$

Moreover we have

$$a_h^+(h_1 \circ \cdots \circ h_n) = h \circ h_1 \circ \cdots \circ h_n, \qquad a_h^- h_1 \circ \cdots \circ h_n = \sum_{i=1}^n \langle h, h_i\rangle h_1 \circ \cdots \circ \not{h}_i \circ \cdots \circ h_n,$$

$$a_h^- \mathcal{E}(k) = \langle h, k\rangle \mathcal{E}(k), \qquad a_h^+ \mathcal{E}(k) = \frac{\mathrm{d}}{\mathrm{d}t}|_{t=0} \mathcal{E}(k + th).$$

▷ **Rigid motions of the Hilbert space.** Denote with G the group of rigid motions of $\mathcal{H}$. The elements of G are pairs $g = (h, U)$ with $h \in \mathcal{H}$ and U a unitary operator on $\mathcal{H}$. The action is $gf := Uf + h$ so that $g' gf = g'(Uf + h) = U'Uf + U'h + h'$ and $(h', U')(h, U) = (U'h + h', U'U)$. Correspondingly we can define extended Weyl operators $W_g = W_{h,U}$ by

$$W_{h,U}\mathcal{E}(u) = e^{-C_{h,U}(u)} \mathcal{E}((h, U)u), \qquad C_{h,U}(u) = \langle h, Uu\rangle + \frac{1}{2}\|h\|^2.$$

Note that the family of functions $(C_g)_{g \in G}$ satisfies

$$C_{g'g}(u) = C_g(g'u) + C_{g'}(u) - i\,\mathrm{Im}\langle U'u, u'\rangle$$

and therefore we have Weyl relations

$$W_{g'} W_g = e^{-i\mathrm{Im}\langle u', U'u\rangle} W_{g'g}.$$

From these relations it follows that W_g is preserves the scalar product and is invertible and then can be extended by continuity to all $\mathcal{F}$ as a unitary operator. Therefore $(W_g)_g$ is a projective unitary representation of G in $\mathcal{F}$. As usual by now we can harvest its strongly continuous one parameter subgroups to obtain random variables.

We will denote by P_h the generator of the pure translations $(W_{(th,I)})_t$ in the direction of h. Moreover if H is bounded and selfadjoint on $\mathcal{H}$ we can consider the strongly continuous unitary group $(W_{(0,U_t)})_t$ where $U_t = e^{iHt}$ and denote by $\lambda(H)$ its generator, a selfadjoint operator on $\mathcal{F}$. This is called the *second quantisation* of H. On the exponential domain $W_{(0,U_t)}\mathcal{E}(u) = \mathcal{E}(U_t u)$ and

$$\partial_t|_{t=0} W_{(0,U_t)}\mathcal{E}(u) = \sum_{n \geqslant 1} \frac{1}{(n-1)!} u^{\circ n-1} \circ Hu = Hu \circ \mathcal{E}(u) = a_{Hu}^+ \mathcal{E}(u).$$

If $(e_n)_n$ is an ONB of $\mathcal{H}$ we have the heuristic formula

$$\lambda(H) = \sum_n a_{He_n}^+ a_{e_n}^-.$$

Therefore we will also denote $\lambda(H) = a^\circ(H)$.

A particular case is when $U_t = e^{it\gamma}$ for some bounded measurable real function $\gamma: \mathbb{R}_+ \to \mathbb{R}$. In this case $Hf = \gamma f$ and we denote $\lambda(H) = a^\circ_\gamma$ whose action on symmetric tensors is given by

$$a^\circ_\gamma(h_1 \circ \cdots \circ h_n) = \gamma h_1 \circ \cdots \circ h_n + h_1 \circ \gamma h_2 \circ \cdots \circ h_n + \cdots + h_1 \circ \cdots \circ \gamma h_n.$$

$\triangleright$ **Some further random variables.** Let $(\alpha(t))_t$, $(h(t))_t$, $(U(t))_t$ be C^1 curves in $\mathbb{C}$, $\mathcal{H}$, and in unitary operators of $\mathcal{H}$ (in strong topology), respectively. Consider

$$Z_t = e^{i\alpha(t)} W_{(h(t), U(t))}$$

and assume that things are such that $(Z_t)_t$ is a strongly continuous one parameter group. Assume $\alpha(0) = 0$, $h(0) = 0$, $U(0) = \mathbb{I}$, and let $\alpha'(0) = \alpha'$, $h'(0) = h'$, $U'(0) = iH$. Then we can compute its generator on the exponential domain $\mathcal{E}$ as follows. Note first that

$$Z_t \mathcal{E}(u) = e^{i\alpha(t)} e^{-\langle h(t), U(t)u\rangle - \frac{1}{2}\|h(t)\|^2} \mathcal{E}(h(t) + U(t)u).$$

Now

$$Z_0' \mathcal{E}(u) = i\alpha' \mathcal{E}(u) - \langle h', u\rangle \mathcal{E}(u) + \frac{\mathrm{d}}{\mathrm{d}t}|_{t=0} \mathcal{E}(h(t) + U(t)u)$$

$$= (i\alpha' - a^-_{h'} + a^+_{h'} + i a^\circ_H) = i(\alpha' - P_{h'} + a^\circ_H)$$

where the reader can check that indeed $P_{h'} = i(a^+_{h'} - a^-_{h'})$.

$\triangleright$ **Enters time.** Let us now specialise our situation to the interesting case $\mathcal{H} = L^2(\mathbb{R}_+)$. In this case we can take $a^\pm_t = a^\pm_{\mathbb{I}_{[0,t]}}$ and observe that the (commutative) family of random variables $(Q_t)_t$ is distributed as a Brownian motion on the vaccuum state. Similarly for $(P_t)_t$.

$\triangleright$ **A Poisson process.** The Fock space $\mathcal{F}$ does not contains only Brownian motion, there is also a Poisson process there. Let $\gamma \in L^2(\mathbb{R}_+)$ and compactly supported. Consider

$$Z(\gamma) = e^{i\alpha(\gamma)} W_{(h(\gamma), U(\gamma))}$$

where $U(\gamma) f(r) = e^{i\gamma(r)} f(r)$, $h(\gamma)(r) = c(e^{i\gamma(r)} - 1) \in L^2(\mathbb{R}_+)$ then

$$Z_\gamma Z_{\gamma'} = e^{i(\alpha(\gamma) + \alpha(\gamma'))} W_{(h(\gamma), U(\gamma))} W_{(h(\gamma'), U(\gamma'))}$$

$$= e^{i(\alpha(\gamma) + \alpha(\gamma'))} W_{(h(\gamma) + U(\gamma)h(\gamma'), U(\gamma)U(\gamma'))} e^{-i\,\mathrm{Im}\langle h(\gamma), U(\gamma)h(\gamma')\rangle}.$$

where $h(\gamma) + U(\gamma)h(\gamma') = c(e^{i\gamma} - 1) + c(e^{i(\gamma+\gamma')} - e^{i\gamma}) = c(e^{i(\gamma+\gamma')} - 1) = h(\gamma + \gamma')$ and

$$\mathrm{Im}\,\langle h(\gamma), U(\gamma)h(\gamma')\rangle = \mathrm{Im} \int_0^\infty c^2 (1 - e^{i\gamma(r)})(e^{i\gamma'(r)} - 1)\mathrm{d}r$$

$$= c^2\, \mathrm{Im} \int_0^\infty \sin(\gamma(r)) + \sin(\gamma'(r)) - \sin(\gamma(r) + \gamma'(r))\mathrm{d}r.$$

Letting $\alpha(\gamma) = c^2 \int_0^\infty \sin(\gamma(r))\mathrm{d}r$ we have

$$Z(\gamma)Z(\gamma') = e^{i\alpha(\gamma+\gamma')} W_{(h(\gamma+\gamma'), U(\gamma+\gamma'))} = Z(\gamma + \gamma') = Z(\gamma')Z(\gamma).$$

Therefore $(Z_t(\gamma) = Z(t\gamma))_t$ is a strongly continuous unitary group whose generator X_γ, by our computations above is $\alpha' - P_{h'} + a^\circ_H$ with $\alpha' = c^2\int_0^\infty \gamma(r)\mathrm{d}r$, $h' = ci\gamma$, $Hf = \gamma f$ and $Z_t(\gamma) = e^{itX_\gamma}$ with

$$X_\gamma = c^2\int_0^\infty \gamma(r)\mathrm{d}r + Q_{c\gamma} + a^\circ_\gamma.$$

Moreover $X_{\gamma+\gamma'} = X_\gamma + X_{\gamma'}$ and $(X_\gamma)_\gamma$ is a commuting family of self–adjoint operators whose law in the vacuum state is given by

$$\langle \Phi, e^{itX_\gamma}\Phi\rangle = \langle \Phi, e^{i\alpha(t)}W_{(h(t),U(t))}\Phi\rangle = e^{i\alpha(t)}e^{-\frac{1}{2}\|h(t)\|^2} = e^{i\,c^2\,\mathrm{Im}\int_0^\infty \sin(t\gamma(r))\mathrm{d}r - \frac{1}{2}\int_0^\infty c^2|e^{it\gamma(r)}-1|^2\mathrm{d}r}$$

$$= e^{i\,c^2\int_0^\infty (e^{it\gamma(r)}-1)\mathrm{d}r}.$$

Moreover if γ, γ' have disjoint support we have

$$\langle \Phi, e^{itX_\gamma}e^{isX_{\gamma'}}\Phi\rangle = \langle \Phi, e^{i\,X_{t\gamma+s\gamma'}}\Phi\rangle$$

$$= e^{ic^2\int_0^\infty (e^{i\,(t\gamma(r)+s\gamma'(r))}-1)\mathrm{d}r} = e^{ic^2\int_0^\infty (e^{it\gamma(r)}-1)\mathrm{d}r}e^{ic^2\int_0^\infty (e^{is\gamma'(r)}-1)\mathrm{d}r}$$

$$= \langle \Phi, e^{itX_\gamma}\Phi\rangle\langle \Phi, e^{isX_{\gamma'}}\Phi\rangle$$

so $X_\gamma, X_{\gamma'}$ are independent. Taking $\gamma(r) = \mathbb{I}_{[0,a]}$ and letting $X_a = X_\gamma$ we have that $(X_a)_{a\geqslant 0}$ is a Poisson process of intensity c^2.