Non-perturbative Renormalization

The Key Lemma

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S4F3 - Graduate Seminar on Applied Probability

The RG map

We consider here $\varepsilon \in \mathbb{C}$ complex.

RG map: It is the map $R(\varepsilon,\gamma)\colon B_{\mathrm{trim}} \to B_{\mathrm{trim}}$ given by

$$\sum_{(\ell_i)_1^n} \mathcal{R}_\ell^{\ell_1,\ldots,\ell_n}(H) = \sum_{(\ell_i)_1^n} R_\ell^{\ell_1,\ldots,\ell_n}(H_{\ell_1},\ldots,H_{\ell_n}), \qquad H \in B_{\mathrm{trim}},$$

and it is a composition between integrating-out, trimming, and dilatation.

Remark

Since the fluctuations propagator and the scaling in the dilatation are both analytic functions w.r.t. ε, then also the RG map will be analytic w.r.t. ε whenever the sum is convergent.

Set for ε and norm bounds

Let T be a compact subset of the half-plane

$$T \subset \{\varepsilon \in \mathbb{C}\colon \operatorname{Re}\varepsilon < d/6\}.$$

Same bound for $S_l^{\ell_1,\ldots,\ell_n}$ with uniform T-dependent constants for $\varepsilon \in T$.

The action of dilatation for complex ε has the same form with $D_l = l[\psi] - d$ complex, since $[\psi] = d/4 - \varepsilon/2$. For its bounds, replace D_l with $\operatorname{Re} D_l$:

$$\|DH_{\ell,p}\|_w = \gamma^{-\operatorname{Re} D_l - p}\|H_{\ell,p}\|_{w(\cdot/\gamma)} \leqslant \gamma^{-\operatorname{Re} D_l - p}\|H_{\ell,p}\|_w, \qquad \varepsilon \in \mathbb{C}.$$

New criterion for irrelevance: $\operatorname{Re} D_l + p > 0$.

The parameter $\overline{D}$ is redefined as

$$\overline{D} = \overline{D}(T) = \frac{1}{2}\min_{\varepsilon \in T}\{\operatorname{Re} D_2(\varepsilon) + 2, \operatorname{Re} D_4(\varepsilon) + 1, \operatorname{Re} D_6(\varepsilon)\} > 0.$$

Bounds on $R_\ell^{\ell_1,\dots,\ell_n}$

Same replacement for the estimates for multilinear maps $R_\ell^{\ell_1,\dots,\ell_n}$.

- If $(n;(\ell_1,\dots,\ell_n))=(1;\ell)$, then

$$\|R_\ell^\ell(H_\ell)\|_w \leqslant \begin{cases} \gamma^{-\mathrm{Re}\,D_2-2}\|H_{2\mathrm{R}}\|_w & \text{if } \ell=2\mathrm{R} \\ \gamma^{-\mathrm{Re}\,D_4-1}\|H_{4\mathrm{R}}\|_w & \text{if } \ell=4\mathrm{R} \\ \gamma^{-\mathrm{Re}\,D_l}\|H_\ell\|_w & \text{if } l=|\ell|\geqslant 6 \end{cases},$$

  while $|R_{2\mathrm{L}}^{2\mathrm{L}}(\nu)|=\gamma^{-\mathrm{Re}\,D_2}|\nu|$ and $|R_{4\mathrm{L}}^{4\mathrm{L}}(\lambda)|=\gamma^{-\mathrm{Re}\,D_4}|\lambda|$.

- If $(n;(\ell_1,\dots,\ell_n))\neq(1;\ell)$, then

$$\|R_\ell^{\ell_1,\dots,\ell_n}(h_1,\dots,h_n)\|_w \leqslant \gamma^{-\mathrm{Re}\,D_l}\rho_l(h_1,\dots,h_n), \qquad h_i\in B_{\ell_i},$$

$$\rho_l(h_1,\dots,h_n):=\begin{cases} C_\gamma^{n-1}\prod_{i=1}^n C_0^{|\ell_i|}\|h_i\|_w & \text{if } \sum_i|\ell_i|\geqslant l+2(n-1) \\ 0 & \text{otherwise} \end{cases}.$$

Definitions of e_ν, e_λ, e_u

Fixed point equation: Denoting $y=(\nu,\lambda,u)$, the FPE reads as

$$\begin{aligned}
\nu &= \gamma^{d/2+\varepsilon}(\nu + l_1\lambda) + e_\nu^{(0)}(y),\\
\lambda &= \gamma^{2\varepsilon}(\lambda + l_2\lambda^2) + e_\lambda^{(0)}(y),\\
u &= e_u(y),
\end{aligned}$$

where

$$\begin{aligned}
e_\nu^{(0)}(y) &= \sum_{(\ell_i)_1^n \neq (2\mathrm{L}),(4\mathrm{L})} R_{2\mathrm{L}}^{\ell_1,\dots,\ell_n}(H_{\ell_1},\dots,H_{\ell_n}),\\
e_\lambda^{(0)}(y) &= \sum_{(\ell_i)_1^n \neq (2\mathrm{L}),(4\mathrm{L},4\mathrm{L}),(6\mathrm{SL})} R_{4\mathrm{L}}^{\ell_1,\dots,\ell_n}(H_{\ell_1},\dots,H_{\ell_n}),\\
e_u(y) &= \sum_{(\ell_i)_1^n} R_{\ell}^{\ell_1,\dots,\ell_n}(H_{\ell_1},\dots,H_{\ell_n}), \qquad \ell \neq 2\mathrm{L}, 4\mathrm{L}, 6\mathrm{SL}.
\end{aligned}$$

Given γ-dependent constants A_0, A_0^R, A_1^R, A_2^R, A, we denote by $\|u\|_{B(\gamma,\delta)}$ the following norm of a vector $u = (H_\ell)_{\ell \neq 2\mathrm{L},4\mathrm{L},6\mathrm{SL}} \equiv (u_\ell)_{\ell \in \{2\mathrm{R},4\mathrm{R},6\mathrm{R},8,10,\dots\}}$ of irrevelant components (6SL exlcuded)

$$\|u\|_{B(\gamma,\delta)} = \max\left\{\frac{\|u_{2\mathrm{R}}\|_w}{A_0^{\mathrm{R}}\delta^2}, \frac{\|u_{4\mathrm{R}}\|_w}{A_1^{\mathrm{R}}\delta^2}, \frac{\|u_{6\mathrm{R}}\|_w}{A_2^{\mathrm{R}}\delta^3}, \sup_{\ell \geqslant 8} \frac{\|u_\ell\|_w}{A\delta^{k(\ell)}}\right\},$$

where $k(\ell) = \ell/2 - 1$, in terms of which the norm of $y = (\nu, \lambda, u)$ can be rewritten

$$\|y\|_{Y(\gamma,\delta)} = \max\left\{\frac{|\nu|}{A_0\delta}, \frac{|\lambda|}{A_0\delta}, \|u\|_{B(\gamma,\delta)}\right\}.$$

Moreover,

$$\mathfrak{X}_*(x) = -8\lambda^2 \sum_{n=1}^{\infty} \gamma^{(2d-6[\psi])n} g(x\gamma^n).$$

Main result

Choose $d \in \{1,2,3\}$, cutoff χ, $N \geqslant 4$, and a compact set $T \subset \mathbb{C}$ as before.

Complex Key Lemma

There exist $\gamma_{\mathsf{key}} \geqslant 2$ and positive continuous functions

$$\delta_0(\gamma), A_0(\gamma), \{A_k^{\mathrm{R}}(\gamma)\}_{k=0,1,2}, A(\gamma), E_0(\gamma), E_1(\gamma), \qquad \text{on } \gamma \geqslant \gamma_{\mathsf{key}},$$

with the following property: Take any $\gamma \geqslant \gamma_{\mathsf{key}}$, any $0 < \delta \leqslant \delta_0(\gamma)$ and any $y = (\nu, \lambda, u)$ satisfying $\|y\|_{Y(\gamma,\delta)} \leqslant 1$, and apply to it the $R(\varepsilon, \gamma)$ with any $\varepsilon \in T$. Then

$$\left|e_\nu^{(0)}(y)\right| \leqslant E_0\delta^2, \qquad \left|e_\lambda^{(0)}(y)\right| \leqslant E_1\delta^3, \qquad \|e_u(y)\|_{B(\gamma,\delta)} \leqslant \gamma^{-\bar{D}},$$

$$\left|\partial_i e_\nu^{(0)}(y)\right| \leqslant \frac{E_0\delta^2}{A_0\delta}, \qquad \left|\partial_i e_\lambda^{(0)}(y)\right| \leqslant \frac{E_1\delta^3}{A_0\delta}, \qquad |\partial_i e_u(y)| \leqslant \frac{\gamma^{-\bar{D}}}{A_0\delta}, \qquad i = \nu, \lambda,$$

$$\left\|\partial_u e_\nu^{(0)}(y)\right\|_{\mathcal{L}(B,\mathbb{R})} \leqslant E_0\delta^2, \qquad \left\|\partial_u e_\lambda^{(0)}(y)\right\|_{\mathcal{L}(B,\mathbb{R})} \leqslant E_1\delta^3, \qquad \|\partial_u e_u(y)\|_{\mathcal{L}(B,B)} \leqslant \gamma^{-\bar{D}},$$

uniformly in $\varepsilon \in T$.

Bounds on e_u: case $\ell \geqslant 8$

We start from the bound on $\|(e_u(y))_\ell\|_w$ with $\ell \geqslant 8$.

Recall

$$(e_u(y))_\ell = \sum_{(\ell_i)_1^n} R_\ell^{\ell_1,\ldots,\ell_n}(H_{\ell_1},\ldots,H_{\ell_n}).$$

Previous bounds:

$$\|(e_u(y))_\ell\|_w \leqslant \gamma^{-\operatorname{Re} D_\ell}\|u_\ell\|_w + \gamma^{-\operatorname{Re} D_\ell}\sum_{(\ell_i)_1^n \neq \ell} \rho_l(H_{\ell_1},\ldots,H_{\ell_n}),$$

where H_{ℓ_i} should be interpreted as equal to

- ν, if $\ell_i = 2\text{L}$,
- λ, if $\ell_i = 4\text{L}$,
- $\mathfrak{X}_*$, if $\ell_i = 6\text{SL}$,
- u_{ℓ_i}, otherwise.

Recall that $\rho_l = 0$ unless $\sum_i |\ell_i| \geqslant l + 2(n-1)$.

Case $\ell \geqslant 8$

By using the assumption $\|y\|_{Y(\gamma,\delta)} \leqslant 1$, we find

$$\begin{aligned}
\|H_{2\mathrm{L}}\|_w + \|H_{2\mathrm{R}}\|_w \leqslant A_0\delta + A_0^R\delta^2 &=: b_0, \\
\|H_{4\mathrm{L}}\|_w + \|H_{4\mathrm{R}}\|_w \leqslant A_0\delta + A_1^R\delta^2 &=: b_1, \\
\|H_{6\mathrm{SL}}\|_w + \|H_{6\mathrm{R}}\|_w \leqslant C_{\gamma 3}A_0^2\delta^2 + A_2^R\delta^3 &=: b_2, \\
\|H_\ell\|_w \leqslant A\delta^{k(\ell)} &=: b_{k(\ell)}, \qquad \text{if } \ell \geqslant 8.
\end{aligned}$$

Recall $k(\ell) = |\ell|/2 - 1$.

Re-arrange the bounds as:

$$b_k \leqslant A\delta^{\max\{k,1\}}, \qquad k \geqslant 0.$$

- For $k \geqslant 3$ this is true as an equality by the definition of b_k.
- To have this for $k = 0, 1, 2$ as well, we will assume (assumptions are marked with ♠):

$$(\spadesuit) \qquad 2\max\left(A_0, A_0^R\delta_0, A_1^R\delta_0, C_{\gamma 3}A_0^2 + A_2^R\delta_0\right) \leqslant A.$$

Case $\ell \geqslant 8$

Recall that

$$\rho_l(H_{\ell_1},\dots,H_{\ell_n}) := \begin{cases} C_\gamma^{n-1}\prod_{i=1}^n C_0^{|\ell_i|}\|H_{\ell_i}\|_w & \text{if } \sum_i |\ell_i| \geqslant l+2(n-1) \\ 0 & \text{otherwise} \end{cases}.$$

We find that

$$\|(e_u(y))_\ell\|_w \leqslant \gamma^{-\operatorname{Re} D_\ell}\big(A\delta^{k(\ell)} + \Delta^{(1)}_{k(\ell)} + \Delta^{(2)}_{k(\ell)}\big),$$

where we defined (here $C = C_0^2$):

$$\Delta^{(1)}_k = \sum_{k'=k+1}^{\infty} C^{k'+1} b_{k'}, \qquad \Delta^{(2)}_k = \sum_{(k_i)_{i=1}^n, n\geqslant 2} F_k[(k_i)_1^n],$$

$$F_k[(k_i)_1^n] = \begin{cases} C_\gamma^{n-1}\prod_{i=1}^n C^{k_i+1} b_{k_i} & \text{if } \sum_i k_i \geqslant k \\ 0 & \text{otherwise} \end{cases}.$$

Case $\ell \geqslant 8$: Estimates on $\Delta_k^{(i)}$

Recall $b_k \leqslant A\delta^{\max\{k,1\}}$, for $k \geqslant 0$, and

$$\Delta_k^{(1)} = \sum_{k'=k+1}^{\infty} C^{k'+1} b_{k'}, \qquad \Delta_k^{(2)} = \sum_{(k_i)_{i=1}^n, n \geqslant 2} F_k[(k_i)_1^n],$$

$$F_k[(k_i)_1^n] = \begin{cases} C_\gamma^{n-1} \prod_{i=1}^n C^{k_i+1} b_{k_i} & \text{if } \sum_i k_i \geqslant k \\ 0 & \text{otherwise} \end{cases}.$$

Lemma

Suppose the nonnegative constants C_γ, C, δ, A satisfy

$$(\spadesuit) \qquad C\delta \leqslant 1/4, \qquad C_\gamma C A \delta \leqslant 1/2, \qquad C_\gamma C A \leqslant 1/2,$$

and that $0 \leqslant b_k \leqslant A\delta^{\max\{k,1\}}$ for all $k \geqslant 0$. Then we have

$$\Delta_k^{(1)} \leqslant A\delta^{k+1}(2C^{k+2}), \qquad \Delta_k^{(2)} \leqslant A\delta^{\max\{k,2\}} \cdot \begin{cases} C_0 = 4C + 8C^2 + 16C^3 & \text{if } k = 0, 1 \\ 2(2C)^{k+1} & \text{if } k \geqslant 2 \end{cases}.$$

Proof of the Lemma

First bound: Easy. We focus on the second bound.

Extending by zeros: For any sequence $\varkappa = (k_i)_1^n$, we say that a sequence $\varkappa'$ *extends* $\varkappa$ *by m zeros* if it is obtained from $\varkappa$ inserting m zeros in arbitrary places.

Define $F_{\text{ext}}[\varkappa]$ as the sum of $F[\varkappa']$ over all $\varkappa'$ extending $\varkappa$ by an arbitrary number of zeros $m \geqslant 0$

$$F_{\text{ext}}[\varkappa] = \sum_{m=0}^{\infty} \sum_{\varkappa'\text{: extends } \varkappa \text{ by } m \text{ zeros}} F[\varkappa'].$$

Since $F[\varkappa'] = F[\varkappa](C_\gamma C b_0)^m$, we obtain

$$F_{\text{ext}}[\varkappa] \leqslant F[\varkappa] \sum_{m=0}^{\infty} \binom{m+n}{m} (C_\gamma C b_0)^m = \frac{1}{(1 - C_\gamma C b_0)^{n+1}} F[\varkappa].$$

Recall $b_0 \leqslant A\delta$. Thus using $C_\gamma C A \delta \leqslant 1/2$ we have

$$F_{\text{ext}}[\varkappa] \leqslant 2^{n(\varkappa)+1} F[\varkappa].$$

Proof of the Lemma

Next, we will sum over sequences with a fixed $\sum k_i$. Namely, we define

$$\Phi_k = \sum_{n=2}^{\infty} \sum_{(k_i)_1^n, \sum k_i = k} F[(k_i)_1^n], \qquad k \geqslant 0.$$

We first estimate Φ_k's and then convert into the estimate for $\Delta_k^{(2)} = \sum_{k'=k}^{\infty} \Phi_{k'}$.

Consider first $k \geqslant 2$. By extending by zeroes, using $b_{k_i} \leqslant A\delta^{k_i}$ and some combinatorics, we arrive to

$$\Phi_k \leqslant \sum_{n=1}^{k} 2^{n+1} \sum_{(k_i)_1^n, \sum k_i = k, k_i \geqslant 1} F[(k_i)_1^n] \leqslant 4C^{k+1} A\delta^k (1 + 2C_\gamma CA)^{k-1},$$

which implies

$$\Phi_k \leqslant 2C(2C\delta)^k A, \qquad k \geqslant 2,$$

once we use $C_\gamma CA \leqslant 1/2$.

With $C_\gamma CA \leqslant 1/2$ one can also show $F_{\text{ext}}[(k_i)_1^n] \leqslant 4C^{k+1} A\delta^{k+m}$.

Some computations for the previous slide:

$$\Phi_k = \sum_{n=2}^{\infty} \sum_{(k_i)_1^n, \sum k_i = k} F[(k_i)_1^n] \leqslant \sum_{n=1}^{k} 2^{n+1} \sum_{(k_i)_1^n, \sum k_i = k, k_i \geqslant 1} F[(k_i)_1^n]$$

Let $(k_i)_1^n, \sum k_i = k,$ with $m = \#\{k_i = 0, i = 1, \ldots, n\}$

$$F[(k_i)_1^n] \leqslant C_\gamma^{n-1} \prod_{i=1}^{n} C^{k_i+1} A \delta^{k_i \vee 1} = C_\gamma^{n-1} C^{\sum_i k_i + n} A^n \delta^{\sum_i k_i + m} \leqslant C_\gamma^{n-1} C^{k+n} A^n \delta^{k+m}.$$

Then, for $k \geqslant 2$, $m = 0$,

$$\Phi_k \leqslant 4 C^{k+1} A \delta^{k+m} \sum_{n=1}^{k} (2 C_\gamma C A)^{n-1} \binom{k-1}{n-1} \leqslant 4 C^{k+1} A \delta^k (1 + 2 C_\gamma C A)^{k-1}.$$

Proof of the Lemma

We bound Φ_k for $k=0,1$.

For $k=0$, Φ_0 involves the sequence $(0,0)$ and its extensions by zeros. We have then

$$\Phi_0 = F_{\text{ext}}[(0,0)] \leqslant 4CA\delta^2.$$

For $k=1$, Φ_1 involves the sequences $(1,0)$, $(0,1)$ and their extensions by zeros. Then

$$\Phi_1 = 2F_{\text{ext}}[(1,0)] \leqslant 8C^2A\delta^2.$$

Estimate $\Delta_k^{(2)} = \sum_{k'=k}^{\infty} \Phi_{k'}$ summing up in geometric progression, which is possible since $2C\delta \leqslant 1/2$, and adding Φ_0 and Φ_1 when needed. We thus obtain

$$\begin{aligned}
\Delta_k^{(2)} &\leqslant C(2C\delta)^k A, \qquad k \geqslant 2, \\
\Delta_1^{(2)} &= \Phi_1 + \Delta_2^{(2)} \leqslant (8C^2 + 16C^3)A\delta^2, \\
\Delta_0^{(2)} &= \Phi_0 + \Phi_1 + \Delta_2^{(2)} \leqslant (4C + 8C^2 + 16C^3)A\delta^2,
\end{aligned}$$

which concludes the proof. □

Case $\ell \geqslant 8$

We find

$$\|(e_u(y))_\ell\|_w \leqslant \gamma^{-\operatorname{Re} D_\ell} A\delta^{k(\ell)}(1 + 2C^{k(\ell)+2}\delta + 2(2C)^{k(\ell)+1}).$$

It follows

$$\|(e_u(y))_\ell\|_w \leqslant \gamma^{-\overline{D}} A\delta^{k(\ell)}, \qquad \ell \geqslant 8,$$

as long as we impose

$$1 + 2C^{k(\ell)+2}\delta + 2(2C)^{k(\ell)+1} \leqslant \gamma^{\operatorname{Re} D_\ell - \overline{D}}.$$

Given the form of this inequality, it is sufficient to check that it holds for $\ell = 8$, and that the l.h.s. grows slower than the r.h.s. as $\ell \to \ell + 2$, which amounts to two requirements:

$$(\spadesuit) \qquad 1 + 2C^4\delta + 2(2C)^4 \leqslant \gamma^{\operatorname{Re} D_8 - \overline{D}}, \qquad \max(1, C, 2C) \leqslant \gamma^{d/2 - \operatorname{Re}\varepsilon}.$$

Next, we estimate derivatives.

Consider a vector $\delta y = (\delta\nu, \delta\lambda, \delta u)$ satisfying $\|\delta y\| \leqslant 1$, and a trimmed coupling sequence δH_ℓ which contains the couplings in δy and, in addition, the coupling $\delta H_{6\mathrm{SL}}$ corresponding to the variation of $\mathfrak{X}_*$.

We have

$$\nabla_y(e_u(y))_\ell\, \delta y = \sum_{(\ell_i)_1^n} \sum_{i=1}^{n} R_\ell^{\ell_1,\dots,\ell_n}(H_{\ell_1},\dots,\delta H_{\ell_i},\dots,H_{\ell_n}),$$

and thus

$$\|\nabla_y(e_u(y))_\ell\, \delta y\|_w \leqslant \gamma^{-\mathrm{Re}\, D_\ell}\|\delta H_\ell\|_w + \gamma^{-\mathrm{Re}\, D_\ell} \sum_{(\ell_i)_1^n \neq \ell} \sum_{i=1}^{n} \rho_l(H_{\ell_1},\dots,\delta H_{\ell_i},\dots,H_{\ell_n}).$$

Note that $\|\delta H_{6\mathrm{SL}}\|_w \leqslant 2C_{3\gamma}A_0^2\delta^2$. We will increase $C_{3\gamma}$ by factor 2.

Then all couplings δH_ℓ satisfies the same bounds as the ones on H_ℓ used to estimate $\|(e_u(y))_\ell\|_w$.

It follows that the functions ρ_l can be estimated in exactly the same way.

Case $\ell \geqslant 8$

This gives

$$\|\nabla_y (e_u(y))_\ell \,\delta y\|_w \leqslant \gamma^{-\operatorname{Re} D_\ell}\big(A\delta^{k(\ell)} + \Delta^{(1)}_{k(\ell)} + \tilde{\Delta}^{(2)}_{k(\ell)}\big),$$

where $\tilde{\Delta}^{(2)}_k$ differs from $\Delta^{(2)}_k$ in that $F_k[(k_i)]$ is replaced by

$$\tilde{F}_k[(k_i)_1^n] = n F_k[(k_i)_1^n],$$

where the factor n accounts for the sum $\sum_{i=1}^n$.

We increase C_γ in by 2 to absorb this factor (note $n \leqslant 2^{n-1}$), so that both $\tilde{F}_k$ and F_k can be considered to satisfy the same bound.

Then, under the same assumptions for the bound on $\|(e_u(y))_\ell\|_w$, we will have

$$\|\nabla_y (e_u(y))_\ell \,\delta y\|_w \leqslant \gamma^{-\overline{D}} A\delta^{k(\ell)}, \qquad \ell \geqslant 8.$$

Taking into account the assumed bounds on couplings δy, this inequality is precisely what is asserted in the Key Lemma concerning the part of e_u with $\ell \geqslant 8$.

Bounds on e_u: case $\ell = 6\mathrm{R}$

Recall $R_{6\mathrm{R}}^{\ell_1,\dots,\ell_n} = DS_6^{\ell_1,\dots,\ell_n}$ if $(\ell_i)_1^n \neq (6\mathrm{SL}), (4\mathrm{L}, 4\mathrm{L})$, and zero otherwise.

From the definitions and the bounds on $R_{6\mathrm{R}}^{\ell_1,\dots,\ell_n}$ we find that

$$\|(e_u(y))_{6\mathrm{R}}\|_w \leqslant \gamma^{-\mathrm{Re}\,D_6}\|u_{6\mathrm{R}}\|_w + \gamma^{-\mathrm{Re}\,D_6} \sum_{(\ell_i)_1^n \neq (6\mathrm{SL}),(6\mathrm{R}),(4\mathrm{L},4\mathrm{L})} \rho_6[(\ell_i)_1^n].$$

By repeating a discussion analogous to that of the previous part, we get

$$\|(e_u(y))_{6\mathrm{R}}\|_w \leqslant \gamma^{-\mathrm{Re}\,D_6}\big(A_2^R\delta^3 + \Delta_2^{(1)} + \Delta_{2;6\mathrm{R}}^{(2)}\big),$$

where $\Delta_{2;6\mathrm{R}}^{(2)}$ is defined analogously to $\Delta_2^{(1)}$: $k(\ell) = |\ell|/2 - 1$

$$\Delta_{2;6\mathrm{R}}^{(2)} = 2C_\gamma C^4 b_1 b_1^R + \sum_{(k_i)_1^n \neq (1,1)}^{n \geqslant 2} F_2[(k_i)_1^n],$$

with $b_1^R = A_1^R\delta^2$. The difference is due to the fact that there is no $(4\mathrm{L}, 4\mathrm{L})$.

Case $\ell = 6\mathrm{R}$

It is convenient to define, for any subsequence $\varkappa = (k_i)_1^n$,

$$F_{\mathrm{ext}}[\varkappa] = \sum_{\varkappa'\text{: extends } \varkappa \text{ by } \geqslant 0 \text{ zeros}} F[\varkappa'].$$

Split the second term in $\Delta^{(2)}_{2;6\mathrm{R}}$ into the contributions of sequences $(1,1,0)$, $(2,0)$, their permutations and extensions by zero and sequences with $\sum k_i \geqslant 3$ which form $\Delta^{(2)}_3$:

$$\Delta^{(2)}_{2;6\mathrm{R}} = 2C_\gamma C^4 b_1 b_1^R + 2F_{\mathrm{ext}}[(2,0)] + 3F_{\mathrm{ext}}[(1,1,0)] + \Delta^{(2)}_3.$$

One can show that, with the same assumptions as for $\ell \geqslant 8$,

$$F_{\mathrm{ext}}[(k_i)_1^n] \leqslant 4C^{k+1}A\delta^{k+m},$$

where $k = \sum k_i$ and m is the number of zeros in the sequence $(k_i)_1^n$.

Case $\ell = 6\mathrm{R}$

We have

$$\Delta^{(2)}_{2;6\mathrm{R}} = 2C_\gamma C^4 b_1 b_1^R + 2F_{\mathrm{ext}}[(2,0)] + 3F_{\mathrm{ext}}[(1,1,0)] + \Delta^{(2)}_3.$$

By the bound on $\Delta^{(2)}_k$ with $k=3$, $b_1^R \leqslant A_1^R \delta^2$, $C_\gamma C A \leqslant 1/2$, and $F_{\mathrm{ext}}[(k_i)_1^n] \leqslant 4C^{k+1} A \delta^{k+m}$ we get

$$\Delta^{(2)}_{2;6\mathrm{R}} \leqslant (C^3 A_1^R + (8C^3 + 12C^3 + 2(2C)^4)A)\delta^3.$$

This implies

$$\|(e_u(y))_{6\mathrm{R}}\|_w \leqslant \gamma^{-\mathrm{Re}\, D_6} \delta^3 (A_2^R + 2C^4 A + C^3 A_1^R + (20C^3 + 2(2C)^4)A),$$

which is smaller than $\gamma^{-\overline{D}} A_2^R \delta^3$, provided that

$$(\spadesuit) \qquad A_2^R + 2C^4 A + C^3 A_1^R + (20C^3 + 2(2C)^4)A \leqslant \gamma^{\mathrm{Re}\, D_6 - \overline{D}} A_2^R.$$

Bounds on e_u: case $\ell = 4\mathrm{R}$

From the definitions and the bounds on $R_{4\mathrm{R}}^{\ell_1,\ldots,\ell_n}$ we find that

$$\|(e_u(y))_{4\mathrm{R}}\|_w \leqslant \gamma^{-\mathrm{Re}\,D_4-1}\|u_{4\mathrm{R}}\|_w + \gamma^{-\mathrm{Re}\,D_4} \sum_{(\ell_i)_1^n \neq (4\mathrm{L}),(4\mathrm{R})} \rho_4[(\ell_i)_1^n],$$

so that

$$\|(e_u(y))_{4\mathrm{R}}\|_w \leqslant \gamma^{-\mathrm{Re}\,D_4-1}\big(A_1^R\delta^2 + \gamma\Delta_1^{(1)} + \gamma\Delta_1^{(2)}\big),$$

which gives

$$\|(e_u(y))_{4\mathrm{R}}\|_w \leqslant \gamma^{-\mathrm{Re}\,D_4-1}\delta^2(A_1^R + \gamma A(2C^3) + \gamma C_0 A).$$

This is smaller than $\gamma^{-\overline{D}}A_1^R\delta^2$, provided that

$$(\spadesuit) \qquad A_1^R + \gamma A(2C^3 + C_0) \leqslant \gamma^{\mathrm{Re}\,D_4+1-\overline{D}}A_1^R.$$

Bounds on e_u: case $\ell = 2R$

From the definitions and the bounds on $R_{2R}^{\ell_1,\ldots,\ell_n}$ we find that

$$\|(e_u(y))_{2R}\|_w \leqslant \gamma^{-\mathrm{Re}\, D_2 - 2}\|u_{2R}\|_w + \gamma^{-\mathrm{Re}\, D_2} \sum_{(\ell_i)_1^n \neq (2L),(2R),(4L)} \rho_2[(\ell_i)_1^n],$$

so that

$$\|(e_u(y))_{2R}\|_w \leqslant \gamma^{-\mathrm{Re}\, D_2 - 2}\big(A_0^R\delta^2 + \gamma^2\Delta_{0;2R}^{(1)} + \gamma^2\Delta_0^{(2)}\big),$$

where

$$\Delta_{0;2R}^{(1)} = C^2 b_1^R + \Delta_1^{(1)} \leqslant C^2 A_1^R\delta^2 + A\delta^2(2C_3^3).$$

Therefore,

$$\|(e_u(y))_{2R}\|_w \leqslant \gamma^{-\mathrm{Re}\, D_2 - 2}\delta^2(A_0^R + \gamma^2(C^2A_1^R + 2C^3A + C_0A)).$$

This is smaller than $\gamma^{-\overline{D}}A_0^R\delta^2$, provided that

$$(\spadesuit) \qquad A_0^R + \gamma^2(C^2A_1^R + 2C^3A + C_0A) \leqslant \gamma^{\mathrm{Re}\, D_2 + 2 - \overline{D}}A_0^R.$$

Bounds on $e_\nu^{(0)}$

$$e_\nu^{(0)}(y) = \sum_{(\ell_i)_1^n \neq (2\mathrm{L}),(4\mathrm{L})} R_{2\mathrm{L}}^{\ell_1,\dots,\ell_n}(H_{\ell_1},\dots,H_{\ell_n}).$$

From the definitions and the bounds on $R_{2\mathrm{L}}^{\ell_1,\dots,\ell_n}$ we find that

$$\left|e_\nu^{(0)}(y)\right| \leqslant \gamma^{-\mathrm{Re}\,D_2} \sum_{(\ell_i)_1^n \neq (2\mathrm{L}),(2\mathrm{R}),(4\mathrm{L})} \rho_2[(\ell_i)_1^n],$$

so that

$$\left|e_\nu^{(0)}(y)\right| \leqslant \gamma^{-\mathrm{Re}\,D_2}\big(\Delta_{0;2\mathrm{R}}^{(1)} + \Delta_0^{(2)}\big),$$

which gives

$$\left|e_\nu^{(0)}(y)\right| \leqslant \gamma^{-\mathrm{Re}\,D_2}\delta^2(C^2A_1^R + 2C^3A + C_0A).$$

We thus get the bound with

$$E_0 = \gamma^{-\mathrm{Re}\,D_2}(C^2A_1^R + 2C^3A + C_0A).$$

Bounds on $e_\lambda^{(0)}$

Recall,

$$e_\lambda^{(0)}(y) = \sum_{(\ell_i)_1^n \neq (4\mathrm{L}),(4\mathrm{L},4\mathrm{L}),(6\mathrm{SL})} R_{4\mathrm{L}}^{\ell_1,\ldots,\ell_n}(H_{\ell_1},\ldots,H_{\ell_n}).$$

From the definitions and the bounds on $R_{4\mathrm{L}}^{\ell_1,\ldots,\ell_n}$ we find that

$$\left|e_\lambda^{(0)}(y)\right| \leqslant \gamma^{-\mathrm{Re}\,D_4} \sum_{\substack{(\ell_i)_1^n \neq (4\mathrm{L}),(4\mathrm{R}),(6\mathrm{SL}),\\ (4\mathrm{L},4\mathrm{L}),(4\mathrm{L},2\mathrm{L}),(4\mathrm{R},2\mathrm{L})}} \rho_4[(\ell_i)_1^n] \leqslant \gamma^{-\mathrm{Re}\,D_4}\left(\Delta_{1;\lambda}^{(1)} + \Delta_{1;\lambda}^{(2)}\right),$$

where

$$\begin{aligned} \Delta_{1;\lambda}^{(1)} &= C^3 b_2^R + \Delta_3^{(1)} \leqslant C^3 A_2^R \delta^3 + A\delta^3(2C^4), \\ \Delta_{1;\lambda}^{(2)} &= 2C_\gamma C^4 b_1 b_1^R + \sum_{(k_i)_1^n \neq (1,1)}^{n\geqslant 2} F_2[(k_i)_1^n] \equiv \Delta_{2;6\mathrm{R}}^{(2)}, \end{aligned}$$

where excluded $(1,0)$, $(1,0,0)$, etc, are excluded from the second term.

Bounds on $e_\lambda^{(0)}$

We see that $\Delta^{(2)}_{1;\lambda}$ is identical to $\Delta^{(2)}_{2;6\mathrm{R}}$ and therefore satisfies the same bound

$$\Delta^{(2)}_{1;\lambda} \leqslant (C^3 A_1^R + (8C^3 + 12C^3 + 2(2C)^4)A)\delta^3.$$

Therefore, we get

$$\left|e_\lambda^{(0)}(y)\right| \leqslant \gamma^{-\operatorname{Re} D_4}\delta^3(C^3 A_2^R + 2C^4 A + C^3 A_1^R + (20C^3 + 2(2C)^4)A).$$

We thus get the the bound, with

$$E_1 = \gamma^{-\operatorname{Re} D_4}(C^3 A_2^R + 2C^4 A + C^3 A_1^R + (20C^3 + 2(2C)^4)A).$$

Tuning the parameters

Finally, we need to show that all the six ♠-constraints can be satisfied consistently.

Recall the constraints:

$$2\max\left(A_0, A_0^R\delta_0, A_1^R\delta_0, C_{\gamma 3}A_0^2 + A_2^R\delta_0\right) \leqslant A,$$

$$C\delta \leqslant 1/4, \qquad C_\gamma CA\delta \leqslant 1/2, \qquad C_\gamma CA \leqslant 1/2,$$

$$1 + 2C^4\delta + 32C^4 \leqslant \gamma^{\mathrm{Re}\,D_8 - \overline{D}}, \qquad \max(1, C, 2C) \leqslant \gamma^{d/2 - \mathrm{Re}\,\varepsilon},$$

$$A_2^R + C^3A_1^R + (20C^3 + 34C^4)A \leqslant \gamma^{\mathrm{Re}\,D_6 - \overline{D}}A_2^R,$$

$$A_1^R + \gamma A(2C^3 + C_0) \leqslant \gamma^{\mathrm{Re}\,D_4 + 1 - \overline{D}}A_1^R,$$

$$A_0^R + \gamma^2(C^2A_1^R + (2C^3 + C_0)A) \leqslant \gamma^{\mathrm{Re}\,D_2 + 2 - \overline{D}}A_0^R.$$

Tuning the parameters

Replace all γ-independent constants in the l.h.s. of the ♠-constraints by their maximum $\overline{C}$.

Also let $\overline{C}_\gamma = \max(C_\gamma, C_{\gamma 3})$ and

$$Z = \min_{\varepsilon \in T} \{\mathrm{Re}\, D_8 - \overline{D}, d/2 - \mathrm{Re}\,\varepsilon, \mathrm{Re}\, D_6 - \overline{D}, \mathrm{Re}\, D_4 + 1 - \overline{D}, \mathrm{Re}\, D_2 + 2 - \overline{D}\} > 0.$$

List of constraints which imply the ♠-constraints for any $\varepsilon \in T$:

$$\overline{C} \leqslant \gamma^Z, \qquad \overline{C}\delta_0 \leqslant 1, \qquad \overline{C}\,\overline{C}_\gamma A \leqslant 1, \tag{♠$_1$}$$

$$\max(A_0, A_0^R\delta_0, A_1^R\delta_0, \overline{C}_\gamma A_0^2 + A_2^R\delta_0) \leqslant \frac{A}{2}, \tag{♠$_2$}$$

$$A_2^R + \overline{C}(A_1^R + A) \leqslant \gamma^Z A_2^R, \quad A_1^R + \overline{C}\gamma A \leqslant \gamma^Z A_1^R, \quad A_0^R + \overline{C}\gamma^2(A_1^R + A) \leqslant \gamma^Z A_0^R. \tag{♠$_3$}$$

The only remaining varying parameter is γ.

We should now choose γ_{key} and δ_0, A_0, $\{A_k^R\}_{k=0,1,2}$, A, E_0, E_1, which are γ-dependent and positive, so that all these constraints hold for $\gamma \geqslant \gamma_{\mathrm{key}}$.

Tuning the parameters

We can satisfy the line (♠$_1$) taking γ large, then A and δ_0 small.

To satisfy line (♠$_3$) we require

$$A_1^R, A \leqslant A_2^R, \qquad \gamma A \leqslant A_1^R, \qquad \gamma^2 A, \gamma^2 A_1^R \leqslant A_0^R,$$

$$1 + 2\overline{C} \leqslant \gamma^Z, \qquad 1 + \overline{C} \leqslant \gamma^Z.$$

The last constraint on γ is of the same type as $\overline{C} \leqslant \gamma^Z$.

Joining the inequalities above and (♠$_2$), the resulting set of constraints reduces to

$$A_0 \leqslant \frac{A}{2}, \qquad \overline{C}_\gamma A_0^2 \leqslant \frac{1}{4} A,$$

$$A_2^R \in \left[A, \frac{1}{4\delta_0} A\right], \qquad A_1^R \in \left[\gamma A, \frac{1}{2\delta_0} A\right], \qquad A_0^R \in \left[\gamma^2 A, \frac{1}{2\delta_0} A\right],$$

$$A_1^R \leqslant A_2^R, \qquad \gamma^2 A_1^R \leqslant A_0^R.$$

Tuning the parameters

Here is then the final order in which all choices have to be made:

- γ_{key} is chosen as the minimal $\gamma \geqslant 2$ satisfying $\bar{C} \leqslant \gamma^Z$ in ($\spadesuit_1$) and $1+\bar{C}, 1+2\bar{C} \leqslant \gamma^Z$.
- We then pick any $\gamma \geqslant \gamma_{\text{key}}$ and compute the constant $\bar{C}_\gamma$.
- We then satisfy $\bar{C}\bar{C}_\gamma A \leqslant 1$ in ($\spadesuit_1$) by choosing $A = (\bar{C}\bar{C}_\gamma)^{-1}$.
- We then choose A_0 sufficiently small to satisfy $A_0 \leqslant A/2$ and $\bar{C}_\gamma A_0^2 \leqslant A/4$.
- Finally, we choose $\delta_0 = \min(\bar{C}^{-1}, 1/(2\gamma^3))$, which satisfies $\bar{C}\delta_0 \leqslant 1$

  $\rightarrow$ All the assumptions in ($\spadesuit_1$) are satisfied.
- At the same time, thanks to $\delta_0 \leqslant 1/(2\gamma^3)$, we can choose A_j^R as follows

$$A_2^R = \frac{1}{4\delta_0}A, \qquad A_1^R = \gamma A, \qquad A_0^R = \frac{1}{2\delta_0}A.$$

  $\rightarrow$ This implies ($\spadesuit_2$) and ($\spadesuit_3$).

This concludes the proof of the Key Lemma.

Appendix: Definition of R_ℓ^ℓ

For $(n;(\ell_1,\ldots,\ell_n))=(1;\ell)$ we have

$$R_\ell^\ell = D,$$

since in this case $S_l^\ell = \mathbf{1}$ and trimming is not needed. In all the other cases $(n;(\ell_1,\ldots,\ell_n))\neq(1;\ell)$, recalling that $l=|\ell|$, we have

$$R_\ell^{\ell_1,\ldots,\ell_n} = D\begin{cases} S_l^{\ell_1,\ldots,\ell_n} & \ell\geqslant 8 \\ T_\ell^l S_l^{\ell_1,\ldots,\ell_n} & \ell\in\{2\mathrm{L},2\mathrm{R},4\mathrm{L},4\mathrm{R}\}\end{cases}$$

$$R_{6\mathrm{SL}}^{\ell_1,\ldots,\ell_n} = D\begin{cases} S_6^{4\mathrm{L},4\mathrm{L}} & (\ell_i)_1^n=(4\mathrm{L},4\mathrm{L}) \\ 0 & \text{otherwise}\end{cases}$$

$$R_{6\mathrm{R}}^{\ell_1,\ldots,\ell_n} = D\begin{cases} S_6^{\ell_1,\ldots,\ell_n} & (\ell_i)_1^n\neq(6\mathrm{SL}),(4\mathrm{L},4\mathrm{L}) \\ 0 & \text{otherwise}\end{cases}$$

where $T_\ell^l\colon B_l\to B_\ell$ is the trimming map.

Just as $S_\ell^{\ell_1,\ldots,\ell_n}$, the map $R_\ell^{\ell_1,\ldots,\ell_n}$ is symmetric and it vanishes unless $\sum_i|\ell_i|\geqslant l+2(n-1)$.