Nonperturbative Renormalization

Finite volume representation and infinite volume limit

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Our goal in this talk is to make sense of the model in finite volume, and give a derivation of the equation

$$H_{\text{eff}}(\mathrm{B}, \mathrm{x}_{\mathrm{B}}) = \mathcal{A} \sum_{n=1}^{\infty} \frac{1}{n!} \sum_{\substack{\mathrm{B}_1, \ldots, \mathrm{B}_n \\ \sum \mathbf{B}_i = \mathbf{B}}} \sum_{\substack{\mathrm{A}_1, \ldots, \mathrm{A}_n \\ \mathrm{A}_i \supset \mathrm{B}_i}} (-)^{\#} \int d^d \mathrm{x}_{\bar{\mathrm{B}}} \mathcal{C}(\mathrm{x}_{\bar{\mathrm{B}}}) \prod_{i=1}^{n} H(A_i, \mathrm{x}_{\mathrm{A}_i})$$

where the sum is over all ways to represent B as a concatenation $\mathrm{B}_1 + \ldots + \mathrm{B}_n$, and then over all ways to extend B_i 's to $\mathrm{A}_i \supset \mathrm{B}_i$. The integration is over points $\mathrm{x}_{\bar{\mathrm{B}}}, \bar{\mathrm{B}} = \bar{\mathrm{B}}_1 + \ldots + \bar{\mathrm{B}}_n, \bar{\mathrm{B}}_i = \mathrm{A}_i \setminus \mathrm{B}_i$. $\mathcal{A}$ denotes the antisymmetrization operation.

Overview:

1. Model in finite volume
2. Effective action in finite volume
3. Norms on Banach space of interactions
4. Effective action in finite volume(Statement and proof idea of the main lemma)

1. Model in finite volume

We consider the model in the region $\mathcal{V} = [-V/2, V/2]^d$ with periodic boundary conditions, so the fields can be expanded into Fourier series

$$\psi_a(x) = \frac{1}{V^d} \sum_{k \in K_V} \psi_{a,k}\, e^{ikx}$$

where $K_V = (2\pi/V)\,\mathbb{Z}^d \cap \operatorname{supp}\chi$ and χ is our UV-cutoff whose $\operatorname{supp}\chi$ is compact hence only finite non-zero terms in the sum. We truncate away all other momenta because they have zero propagator. The coefficients of the Fourier expansion are Grassmann variables.

Then finite volume gaussian measure $d\mu_{P,V}$ is a finite-dimensional measure over Grassmann variables $\psi_{a,k}$

$$d\,\mu_{P,V}(\psi) = \mathrm{Pf}^{-1} \prod_{\psi_{a,k}, k\in K_V} d\,\psi_{a,k}\, e^{S_{2,V}(\psi)}$$

where

$$S_{2,V} = \frac{1}{2\,V^d} \sum_{k\in K_V} \hat{P}(k)^{-1}\,\Omega_{ab}\,\psi_{a,k}\,\psi_{b,-k}$$

where the normalization factor $\mathrm{Pf} > 0$ is the Pfaffian of $S_{2,V}$.

This is a meaningful finite volume version of the formal Gaussian Grassmann measure:

$$d\,\mu_P(\psi) = D\,\psi\,e^{S_2(\psi)}$$

$$S_2(\psi) = \frac{1}{2}\int \frac{d^d\,k}{(2\,\pi)^d}\,\hat{P}(k)^{-1}\,\Omega_{ab}\,\psi_a(k)\,\psi_b\,(-k)$$

Then we can compute the propagator in this finite volume setting

$$\langle \psi_a(x)\, \psi_b(y) \rangle = \Omega_{ab}\, P_V\,(x - y)$$

where

$$P_V(x) = \frac{1}{V^d} \sum_{k \in (2\pi/V)\mathbb{Z}^d} \hat{P}(k)\, e^{ikx}\,, \text{with } \hat{P}(k) = \frac{\chi(k)}{|k|^{\frac{d}{2}+\varepsilon}}$$

For fixed x and $V \to \infty$, we have $P_V(x) \to P(x)$, since a Gaussian measure is completely determined by it correlation function, in this sense we can say that $d\,\mu_{P,V} \to d\,\mu_P$.

The interacting Grassmann measure is then defined as

$$Z_V^{-1}\, d\,\mu_{P,V}(\psi)\, e^{sH_V(\psi)}$$

where

$$H_V(\psi) = \sum_{\mathbf{A}} \int_{\mathcal{V}^l} d^d\mathbf{x}\, H_V(\mathbf{A},\mathbf{x})\, \Psi(\mathbf{A},\mathbf{x})$$

and

$$Z_V = \int d\,\mu_{P,V}(\psi)\, e^{sH_V(\psi)} \text{ (partition function)}$$

Here the factor s is for further convenience, eventually we will set $s = 1$. The model is well defined when $Z_V \neq 0$. They are polynomials in s.(Thus analytic)

2. Effective action in finite volume

In the first talk we talked about the Integrating-out, we split the field ψ as $\psi = \psi_\gamma + \phi$ where ψ_γ is the low-momentum component of ψ, we also split the Grassmann propagator as

$$P(x) = P_\gamma(x) + g(x), \quad \hat{P}_\gamma(k) = \frac{\chi(\gamma k)}{|k|^{\frac{d}{2}+\varepsilon}}, \quad \hat{g}(k) = \frac{\chi(k) - \chi(\gamma k)}{|k|^{\frac{d}{2}+\varepsilon}}$$

and then definine the efective interaction by eliminating ϕ,

$$e^{H_{\text{eff}}(\psi_\gamma)} = \int d\,\mu_g(\phi)\, e^{H(\psi_\gamma + \phi)}$$

Now we try to do it in finite volume.

Define $d\,\mu_{g,V}(\phi)$ in the same way as $d\,\mu_{P,V}(\psi)$ and then consider function

$$I(s,\psi) = \int d\,\mu_{g,V}(\phi)\, e^{sH_V(\psi+\phi)}$$

we see that $I(s,\psi) = e^{sH_V(\psi)}\, p(s,\psi)$ where $p(s,\psi)$ is a polynomial in s, by the special form of H_V and the rules of Grassman calculus.

As we see in the first talk we hope to find $H^V_{\text{eff}}(s,\psi)$ such that

$$e^{H^V_{\text{eff}}(s,\psi)} = I(s,\psi)$$

We define $H^V_{\text{eff}}(s,\psi)$ by the perturbative expansion as we did in the second talk

$$H^V_{\text{eff}}(s,\psi) = \sum_{\text{B}} \int_{\mathcal{V}^{|\text{B}|}} d^d\mathbf{x}\, H^V_{\text{eff}}(s,\mathbf{B},\text{x}_{\text{B}})\, \Psi(\mathbf{B},\text{x}_{\text{B}})$$

where $H^V_{\text{eff}}(s,\mathbf{B},\text{x}_{\text{B}})$ is given by

$$H^V_{\text{eff}}(s,\mathbf{B},\mathbf{x_B}) = \mathcal{A}\sum_{n=1}^{\infty}\frac{s^n}{n!}\sum_{\sum\mathbf{B}_i=\mathbf{B},\mathbf{A}_i\supset\mathbf{B}_i}(-)^{\#}\int_{\mathcal{V}^{|\bar{\mathbf{B}}|}} d^d\mathbf{x}_{\bar{\mathbf{B}}}\,\mathcal{C}_V(\mathbf{x}_{\bar{\mathbf{B}}})\prod_{i=1}^{n}H_V(\mathbf{A}_i,\mathbf{x}_{\mathbf{A}_i})$$

here $\mathcal{C}_V$ is the connected expectation with finite-volume propagtors

$$\mathcal{C}_V(\mathbf{x}_{\bar{\mathbf{B}}}) = \langle \Phi(\bar{\mathbf{B}}_1, \mathbf{x}_{\bar{\mathbf{B}}_1}); \ldots; \Phi(\bar{\mathbf{B}}_n, \mathbf{x}_{\bar{\mathbf{B}}_n}) \rangle_c$$

We claim that the so defined $H^V_{\text{eff}}(s, \psi)$ satisfies $e^{H^V_{\text{eff}}(s,\psi)} = I(s, \psi)$, this is because:

1. The series of $H^V_{\text{eff}}(s, \mathbf{B}, \mathrm{x_B})$ converges and defines $H^V_{\text{eff}}(s, \mathbf{B}, \mathbf{x_B})$ as analytic L_1 -valued functions in the disk $|s| < 2$ (from main lemma below).

2. Since, by perturbation theory, $e^{H^V_{\text{eff}}(s,\psi)}$ and $I(s, \psi)$ have the same Taylor series in s, we conclude $e^{H^V_{\text{eff}}(s,\psi)} = I(s, \psi)$ is satisfied in the disk $|s| < 2$ where they are both analytic, in particular at $s = 1$.

3. This concludes that above formula for $H^V_{\text{eff}}(s, \mathbf{B}, \mathrm{x_B})$ gives the correct effective action in finite volume.

3. Norms on Banach space of interactions

First let me remind you the notations

$$\Psi_A = \begin{cases} \psi_a, & A = a \\ \partial_\mu \psi_a, & A = (a, \mu) \end{cases}, \quad \Psi(\mathrm{A}, \mathrm{x}) = \prod_{i=1}^{l} \Psi_{A_i}(x_i)$$

where $\mathbf{A} = (A_1, \ldots, A_l)$ and $\mathbf{x} = (x_1, \ldots, x_l)$ are finite sequences. $|\mathrm{A}|$ will denote the length of A, and $d(A)$ the number of derivative fields in $\Psi(\mathrm{A}, \mathrm{x})$.

An interaction $H(\psi)$ is a sum of terms with some kernels $H(\mathrm{A}, \mathrm{x})$:

$$H(\psi) = \sum_{\mathrm{A}} \int d^d\mathbf{x}\, H(\mathbf{A}, \mathbf{x})\, \Psi(\mathbf{A}, \mathbf{x})$$

where the number of each $|\mathrm{A}|$ is even, each $H(\mathrm{A}, \mathrm{x})$ is antisymmetric by the Grassman nature of the fields.

Examples are the local quadratic and quartic interactions in the first talk.

We can divide the kernels into groups(couplings) according to the length $|\mathrm{A}|$ (which is also the number of the legs in Feynman diagram):

$$H_l = \{H(A, \mathrm{x})\}_{|\mathrm{A}|=l}$$

For example the local quadratic interaction is in H_2 and local quartic interaction is in H_4.

We will study these groups more carefully in later talks.

In order to study the infinite volume limit, we need norms on these groups of coupling H_l. The size of interaction kernels is measured by means of the weighted L_1 norm:

$$\|H(\mathbf{A})\|_w = \int_{x_1=0} d^d\mathbf{x} |H(\mathbf{A},\mathbf{x})| \, w(\mathbf{x})$$

where $w(\mathbf{x})$ is a translationally invariant weight function, by the translation invariance property, here we perform the integral fixing one of the x coordinates to zero. We also define

$$|H_l|_w = \max_{|\mathbf{A}|=l} \|H(\mathbf{A})\|_w$$

We have to choose suitable $w(\mathbf{x})$ for our situation.

We should incorporate the information about the decay of the kernels $H(\mathbf{A}, \mathbf{x})$ induced by the decay of propagator, whose decay property is induced from UV-cutoff function $\chi \in G^s, s > 1$(Gevrey class).

For propagator P induced from UV-cutoff function $\chi \in G^s, s > 1$(Gevrey class), we know the following inequality:

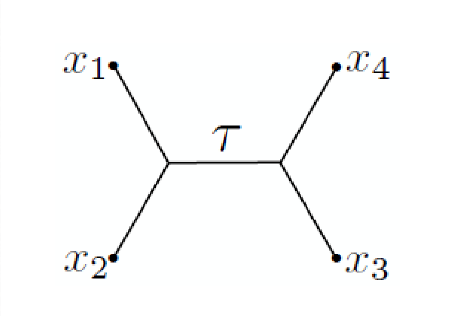
$$|P(x)|, |\partial_\mu P(x)|, |\partial_\mu \partial_\nu P(x)| \leqslant M(x) \equiv C_{\chi 1}\, e^{-C_{\chi 2}|x/\gamma|^\sigma} \quad (x \in \mathbb{R}^d)$$

where $\sigma = 1/s < 1$. The constants $C_{\chi 1}$, $C_{\chi 2}$ depend on χ but are independent of γ. Our kernels are expected to decay with the same rate.

To incorporate the information about the decay of the kernels $H(\mathbf{A},\mathbf{x})$, we choose $w(\mathbf{x})$ growing with a similar rate. A convenient choice turns out to be

$$w(\mathbf{x}) = e^{\bar{C}(\mathrm{St}(\mathbf{x})/\gamma)^{\sigma}}$$

where $\mathrm{St}(\mathrm{x})$ is the Steiner diameter of the set x, it is defined as the length of the shortest tree τ connecting the points in x, and let $\bar{C} = \frac{1}{2} C_{\chi 2}$.



4. Statement and proof of the main lemma

For any kernel $H(\mathbf{A}, \mathbf{x})$ we consider corresponding finite-volume interaction with kernels given by periodization

$$H_V(\mathbf{A}, (0, x_2, \ldots, x_l)) = \sum_{r_i \in \mathbb{Z}^d, i=2\ldots l} H(\mathbf{A}, (0, x_2 + r_2 V, \ldots, x_l + r_l V))$$

Lemma 1. *If there exists $A > 0$ and $\delta > 0$ such that, for any infinite volume interaction satisfying*

$$|H_l|_w \leqslant A\,\delta^{\min(1, l/2-1)} \qquad (l \geqslant 2)$$

and defining the finite volume interactions by periodization for any $V \geqslant 1$, we have

(a) the kernels of H_{eff} and of H^V_{eff} with $s = 1$ are well defined (the series is convergent in L_1);

(b) the kernels of $H^V_{\mathit{eff}}(s)$ are well defined and analytic L_1 -valued functions in the disk $|s| < 2$;

(c) for any B we have $H^V_{\mathrm{eff}}(B, \mathrm{x}) \rightarrow H_{\mathrm{eff}}(\mathrm{B}, \mathrm{x})$ as $V \rightarrow \infty$ in the sense of L_1 norm on any fixed bounded subset of $(\mathbb{R}^d)^l$.

We see that claim (a) for H_{eff}^{V} is a consequence of (b).

For (b), the idea is to bound the L_1 norm for n-th term of the series for H_{eff}^{V}, which is

$$\sum_{\sum \mathbf{B}_i = \mathbf{B}, \mathbf{A}_i \supset \mathbf{B}_i} \frac{|s|^n}{n!} \int_{\mathcal{V}|\mathbf{A}|, x_1 = 0} d^d \mathbf{x}_{\mathbf{A}} |\mathcal{C}_V(\mathbf{x}_{\bar{\mathbf{B}}})| \prod_{i=1}^{n} |H_V(\mathbf{A}_i, \mathbf{x}_{\mathbf{A}_i})|$$

where

$$\mathrm{A} = \mathrm{A}_1 + \cdots + \mathrm{A}_n$$

$$\mathcal{C}_V(\mathbf{x}_{\bar{\mathbf{B}}}) = \langle \Phi(\bar{\mathbf{B}}_1, \mathbf{x}_{\bar{\mathbf{B}}_1}); \ldots; \Phi(\bar{\mathbf{B}}_n, \mathbf{x}_{\bar{\mathbf{B}}_n}) \rangle_c$$

which is defined by propagator g_V.

Here g_V is periodization of g

$$g_V(x) = \frac{1}{V^d} \sum_{k \in (2\pi/V)\mathbb{Z}^d} \hat{g}(k)\, e^{ikx} = \sum_{r \in \mathbb{Z}^d} g\,(x + r\,V)$$

A few facts:

1. By the Gevrey class property of g_V and GKL bound:

$$|\mathcal{C}_V(\mathrm{x}_{\bar{\mathrm{B}}})| \leqslant (C_{\mathrm{GH},V})^{\frac{1}{2}\sum_i l_i} \sum_{\mathcal{T}} \prod_{(xx') \in \mathcal{T}} M_V\,(x - x')$$

where $|A_i| = l_i$, $\mathcal{T}$ is the set of all anchored trees.

2.

$$\int_{\mathcal{V}|\mathbf{A}|,x_1=0} d^d\mathbf{x}_{\mathbf{A}} |\mathcal{C}_V(\mathbf{x}_{\bar{\mathbf{B}}})| \prod_{i=1}^{n} |H_V(\mathbf{A}_i, \mathbf{x}_{\mathbf{A}_i})|$$

$$\overset{\text{by 1.}}{\leqslant} (C_{\mathrm{GH},V})^{\frac{1}{2}\sum_i l_i} \sum_{\mathcal{T}} \int_{\mathcal{V}|A|,x_1=0} d^d\mathbf{x}_{\mathrm{A}} \prod_{(xx')\in\mathcal{T}} M_V(x-x') \prod_{i=1}^{n} |H_V(\mathbf{A}_i, \mathbf{x}_{\mathbf{A}_i})|$$

$$\overset{\text{integration bound}}{\leqslant} (C_{\mathrm{GH},V})^{\frac{1}{2}\sum_i l_i} N_{\mathcal{T}} |M_V|_1^{n-1} \prod_{i=1}^{n} |H_{V,l_i}|_1$$

$$\overset{H_V \text{ is the periodization of } H}{\leqslant} (C_{\mathrm{GH},V})^{\frac{1}{2}\sum_i l_i} N_{\mathcal{T}} |M_V|_1^{n-1} \prod_{i=1}^{n} |H_{l_i}|_w$$

3. Since

$$|H_l|_w \leqslant A\,\delta^{\min(1,l/2-1)} \quad (l \geqslant 2) \text{ and } \mathcal{N}_{\mathcal{T}} \leqslant n! 4^{\sum l_i}$$

then

$$\sum_{\sum \mathbf{B}_i = \mathbf{B}, \mathbf{A}_i \supset \mathbf{B}_i} \frac{|s|^n}{n!} \int_{\mathcal{V}^{|\mathbf{A}|}, x_1 = 0} d^d \mathbf{x}_{\mathbf{A}} |\mathcal{C}_V(\mathbf{x}_{\bar{\mathbf{B}}})| \prod_{i=1}^{n} |H_V(\mathbf{A}_i, \mathbf{x}_{\mathbf{A}_i})|$$

$$\leqslant |s|^n A^n |M_V|_1^{n-1} \sum_{(l_i)_1^n} \prod_{i=1}^{n} (16\, N\, (d+1)\, C_{\mathrm{GH},V}^{1/2})^{\sum l_i} \delta^{\min(1,l_i/2-1)}$$

$$\leqslant \left(\frac{C |s| \delta}{1 - C\delta} \right)^n$$

The last inequality partly comes from uniformly boundedness of $16\,N\,(d+1)\,C_{\mathrm{GH},V}^{1/2}$ and $|M_V|_1^{n-1}$ with respect to V.

See appendix E and F for various estimations.

For part (c),

Fix a bounded subset of $(\mathbb{R}^d)^l$, W.L.O.G, we assume to be centered in the origin, and we call it $\mathcal{V}_0$. We need to show

$$\mathcal{A}\sum_{n=1}^{\infty}\sum_{\sum\mathbf{B}_i=\mathbf{B},\mathbf{A}_i\supset\mathbf{B}_i}\frac{1}{n!}\int_{\mathcal{V}_0^{|\mathbf{B}|}}d^d\mathbf{x}_{\mathbf{B}}|\int_{\mathcal{V}|\bar{\mathbf{B}}|}d^d\mathbf{x}_{\bar{\mathbf{B}}}\mathcal{C}_V(\mathbf{x}_{\bar{\mathbf{B}}})\prod_{i=1}^{n}H_V(\mathbf{A}_i,\mathbf{x}_{\mathbf{A}_i})-$$

$$\int_{\mathbb{R}^{|\bar{\boldsymbol{B}}|d}}d^d\mathbf{x}_{\bar{\mathbf{B}}}\mathcal{C}(\mathbf{x}_{\bar{\mathbf{B}}})\prod_{i=1}^{n}H(\mathbf{A}_i,\mathbf{x}_{\mathbf{A}_i})|$$

tends to 0 as V $\to\infty$. The idea is to bound each n-th term by spliting the integral. We multiply the integrand in absolute value of integral w.r.t. $\mathcal{V}|\bar{\mathrm{B}}|$ by

$$1 = 1\,(\mathrm{St}_V(\mathrm{x_A}) \leqslant V/4) + 1\,(\mathrm{St}_V(\mathrm{x_A}) > V/4)$$

where finite volume Steiner diameter $\mathrm{St}_{\mathsf{V}}(\mathbf{x})$ is the length of the shortest tree on the torus which connects all points in $\mathbf{x} = (x_1, \ldots, x_l)$, clearly we have the bound

$$\mathrm{St}_V(\mathrm{x}) \leqslant \min_{\boldsymbol{r} \in \mathbb{Z}^{dl}} \mathrm{St}(\mathrm{x} + \mathrm{r}V)$$

and do the same for the integral over $\mathbb{R}^{|\bar{\mathbf{B}}|d}$ with standard Steiner diameter.

Thus we can bound each n-th term in the above summation by $|\mathcal{V}_0|\left(R_{1,n}+R_{2,n}+R_{3,n}\right)$ where, $R_{1,n}$,$R_{2,n}$,$R_{3,n}$ are

$$\sum_{\sum \mathbf{B}_i=\mathbf{B},\mathbf{A}_i\supset\mathbf{B}_i}\frac{1}{n!}\int_{\mathcal{V}|\mathbf{A}|,x_1=0}d^d\mathbf{x}_\mathbf{A}|\mathcal{C}_V(\mathbf{x}_{\bar{\mathbf{B}}})|\prod_{i=1}^{n}|H_V(\mathbf{A}_i,\mathbf{x}_{\mathbf{A}_i})|\left(\mathrm{St}_V(\mathbf{x}_\mathbf{A})>V/4\right)$$

$$\sum_{\sum \mathbf{B}_i=\mathbf{B},\mathbf{A}_i\supset\mathbf{B}_i}\frac{1}{n!}\int_{\mathbb{R}^{|\mathbf{A}|d},x_1=0}d^d\mathbf{x}_\mathbf{A}|\mathcal{C}_V(\mathbf{x}_{\bar{\mathbf{B}}})|\prod_{i=1}^{n}|H_V(\mathbf{A}_i,\mathbf{x}_{\mathbf{A}_i})|\,1\left(\mathrm{St}(\mathbf{x}_\mathbf{A})>V/4\right)$$

$$\sum_{\sum \mathbf{B}_i=\mathbf{B},\mathbf{A}_i\supset\mathbf{B}_i}\frac{1}{n!}\int_{\mathbb{R}^{|A|d},x_1=0}d^d\mathbf{x}_\mathrm{A}\left|\mathcal{C}_V(\mathrm{x}_{\bar{\mathrm{B}}})\prod_{i=1}^{n}H_V(\,A_i,\mathrm{x}_{\mathrm{A}_i})-\mathcal{C}(\mathrm{x}_{\bar{\mathrm{B}}})\prod_{i=1}^{n}H(\,A_i,\mathrm{x}_{\mathrm{A}_i})\right|1\left(\mathrm{St}(\mathrm{x}_\mathrm{A})\leqslant V/4\right)$$

In fact all these three parts summation over n are bounded by functions of V tends to 0 exponentially fast.

Facts:

1. With the same method in part (b), one can get a bound for $R_{1,n}$ in the form

$$R_{1,n} \leqslant e^{-C_w (V/(4\gamma))^\sigma} \left(\frac{C\delta}{1 - C\delta} \right)^n$$

similar method and bound can be deduced for $R_{2,n}$.

2. To bound the $R_{3,n}$. The idea is to write the integrand

$$\mathcal{C}_V(\mathrm{x}_{\bar{\mathrm{B}}}) \prod_{i=1}^{n} H_V(A_i, \mathrm{x}_{\mathrm{A}_i}) - \mathcal{C}(\mathrm{x}_{\bar{\mathrm{B}}}) \prod_{i=1}^{n} H(A_i, \mathrm{x}_{\mathrm{A}_i})$$

in telescopic form as the sum of n + 1 terms, like

$$\mathcal{C}_V(\mathrm{x}_{\bar{\mathrm{B}}}) \prod_{i=2}^{n} H_V(\,A_i, \mathrm{x}_{\mathrm{A}_i})(H_V(\,A_1, \mathrm{x}_{\mathrm{A}_1}) - H(\,A_1, \mathrm{x}_{\mathrm{A}_1})) +$$

$$\mathcal{C}_V(\mathrm{x}_{\bar{\mathrm{B}}}) H(\,A_1, \mathrm{x}_{\mathrm{A}_1}) \prod_{i=3}^{n} H_V(\,A_i, \mathrm{x}_{\mathrm{A}_i})(H_V(\,A_2, \mathrm{x}_{\mathrm{A}_2}) - H(\,A_2, \mathrm{x}_{\mathrm{A}_2})) + \cdots$$

in each of which either a difference $\mathcal{C}_V(\mathrm{x}_{\bar{\mathrm{B}}}) - \mathcal{C}(\mathrm{x}_{\bar{\mathrm{B}}})$ or $H_V(\mathbf{A}_i, \mathbf{x}_{\mathbf{A}_i}) - H(\mathbf{A}_i, \mathbf{x}_{\mathbf{A}_i})$, the integral involve $H_V(\mathbf{A}_i, \mathbf{x}_{\mathbf{A}_i}) - H(\mathbf{A}_i, \mathbf{x}_{\mathbf{A}_i})$ part can be bounded by a function of V with from

$$A\,\delta^{\min(1,l/2-1)} e^{-C_w(V/(2\gamma))^\sigma}$$

For $\mathcal{C}_V(\mathrm{x}_{\bar{\mathrm{B}}}) - \mathcal{C}(\mathrm{x}_{\bar{\mathrm{B}}})$, use BBF formula

$$\mathcal{C}(\mathbf{x}_{\bar{\mathbf{B}}}) = \sum_{\mathcal{T}} \prod_{(xx')\in\mathcal{T}} g\,(x - x') \int d\,\mu_T(\mathbf{r}) \det\mathcal{N}$$

$$\mathcal{C}_V(\mathrm{x}_{\bar{\mathrm{B}}}) = \sum_{\mathcal{T}} \prod_{(xx')\in\mathcal{T}} g_V\,(x - x') \int d\,\mu_T(\mathbf{r}) \det\mathcal{N}_V$$

where $\mathcal{N} = \mathcal{N}(r)$ is a Gram matrix .

Again write the difference into the telescopic form and bound involving term $g_V(x - x') - g(x - x')$ or $\det\mathcal{N}_V - \det\mathcal{N}$ by exponentially decay functions of V(also exponentially decay after sum up index n).

So the convergence in L_1 is true.