The RG approach in classical PDE theory

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Seminar - Nonperturbative renormalization group

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Applications of RG approach in other areas

This talk is based on the first of two papers:

- Bricmont, Kupiainen, Lin: *Renormalization group and asymptotics of solutions of nonlinear parabolic equations*, Comm. Pure Appl. Math. (1994)
- Bricmont, Kupiainen: *Renormalization group and the Ginzburg-Landau equation*, Comm. Math. Phys. (1992)

developed with the aim of showing the relevance of the RG approach in other branches of mathematics outside of QFT, in particular **the asymptotic behaviour of nonlinear PDEs**.

For other applications see also:

- Bricmont, Gawedzki, Kupiainen: *KAM theorem and quantum field theory*, Comm. Math. Phys. (1999)
- Li, Sinai: *Blow ups of complex solutions of the 3D Navier–Stokes system and renormalization group method*, JEMS (2008)
- Kupiainen: *Renormalization group and stochastic PDEs*, Ann. H. Poincaré (2016)

Our main goal

Consider for a 1-dimensional PDE of the form

$$\partial_t u = \partial_x^2 u + F(u, \partial_x u, \partial_x^2 u)$$

with initial time $t = 1$. Reference example: **heat equation with absorption**

$$\partial_t u = \partial_x^2 u - u^p, \quad p \geq 0.$$

We are interested in the asymptotics of u of the form

$$u(x,t) \sim t^{-\frac{\alpha}{2}} f^*(t^{-\frac{1}{2}} x) \quad \text{as } t \to \infty.$$

Classical approach consists of:

1) finding a **scale-invariant solution** (assuming F is scale-invariant), which reduces to solving an ODE for f^*;
2) establishing stability of such solution (exploiting scale invariance of the PDE).

Drawbacks: it requires to deal with positive solutions $u \geq 0$; stability often involves the maximum principle, not suitable if ∂_x^2 is replaced by fractional Laplacian $-(-\Delta)^{\beta/2}$; not suited for F not scale-invariant.

The renormalization group approach

The RG method transforms the problem of the large time limit into an **iteration of a fixed time problem followed by a scaling transformation**.

Given F and a solution u to the PDE, we define for some $\alpha \geq 0, L > 1$ to be chosen later the **scaling operator**

$$u_L(t,x) = L^\alpha u(L^2 t, Lx);$$

observe that if u solves the PDE for F, then u_L solves

$$\partial_t u_L = \partial_x^2 u_L + F_L(u_L, \partial_x u_L, \partial_x^2 u_L)$$

where now $F_L(a,b,c) := L^{2+\alpha} F(L^{-\alpha} a, L^{-1-\alpha} b, L^{-2-\alpha} c)$.

Let us assume we are given a Banach space $\mathcal{S}$ of initial data to the PDE which is invariant under the action of the RG map

$$R\, f = R_{L,F}\, f := u_L(x,1) = L^\alpha u(L^2, Lx),$$

where u denotes the solution with $u(1,x) = f(x)$; in general the fact that R maps $\mathcal{S}$ into itself is non trivial and must be checked rigorously.

The renormalization group approach - II

By the correspondence between u_L and F_L, we have the "**semigroup property**"

$$\underbrace{R_{L^n,F}}_{\text{solve on }[1,L^n]} = \underbrace{R_{L,F_{L^{n-1}}}}_{\text{solve on }[L^{n-1},L^n]} \circ \cdots \circ \underbrace{R_{L,F_L}}_{\text{solve on }[L,L^2]} \circ \underbrace{R_{L,F}}_{\text{solve on }[1,L]} .$$

Each R on the r.h.s. involves solving the problem on a finite interval $[1, L]$ and the long time problem is reduced to an iteration of these: setting $t = L^{2n}$, one has

$$u(t,x) = t^{-\alpha/2}(R_{L^n,F}\, f)(t^{-1/2}\, x).$$

The **RG analysis** then consists in showing that there exists α such that

$$F_{L^n} \to F^*, \quad R_{L^n,F} f \to f^*$$

where f^* is a **fixed point of the RG**, corresponding to a scale-invariant solution of $\partial_t u = \partial_x^2 u + F^*(u)$. The asymptotics of the original PDE are then given by

$$u(t, t^{1/2}x) \sim t^{-\alpha/2} f^*(x).$$

Irrelevant and marginal nonlinearities

For $F = 0$, the PDE is just $\partial_t u = \partial_x^2 u$, which has a scale-invariant family of solutions given the **heat kernel** for $\alpha = 1$; for suitable $F \neq 0$ and integrable f, we may expect the rescaled solution u to be attracted by this line of fixed points.

Therefore we set $\alpha = 1$, $F_L(a, b, c) = L^3 F(L^{-1}a, L^{-2}b, L^{-3}c)$; we will assume $F : \mathbb{C}^3 \to \mathbb{C}$ to be analytic in a neighbourhood of the origin.

Observe that for a monomial $F(a, b, c) = a^{n_1} b^{n_2} c^{n_3}$ we have

$$F_L = L^{-d_F} F \quad \text{for } d_F = n_1 + 2n_2 + 3n_3 - 3;$$

for general F we define its **degree** d_F to be the smallest number computed for the monomials in the Taylor series of F at 0 with non-zero coefficient.

We say that F is **irrelevant** if $d_F > 0$, **marginal** if $d_F = 0$, **relevant** if $d_F < 0$.

In this talk we will only deal with the irrelevant and marginal cases; observe that F is irrelevant iff $F_{L^n} \to F^* \equiv 0$.

Gaussian fixed points and suitable Banach spaces

We start by discussing the RG map R_0 corresponding to $F \equiv 0$. We have

$$R_0 f := R_{0,L} f = L\Big(e^{(L^2-1)\partial_x^2} f\Big)(L\,\cdot) = e^{(1-L^{-2})\partial_x^2} f_L \quad \text{for } f_L(x) = Lf(Lx);$$

in Fourier space

$$\widehat{R_0 f}(k) = e^{-k^2(1-L^{-2})}\hat{f}(L^{-1}k).$$

R_0 has a line of fixed points: $\{A\, f_0\}_{A\in\mathbb{R}}$ for f_0^* given by $\hat{f}_0^*(k) = e^{-k^2}$.

Let us now define $\mathcal{S}$ to be the completion of C_c^∞ under

$$\|f\| := \sup_k \big\{(1+k^4)(|\hat{f}(k)| + |\hat{f}'(k)|)\big\}.$$

Reasons for this choice:

1) R_0 has nice contractivity properties w.r.t. $\|\cdot\|$;
2) $\|f\|$ controls norms like $\|f\|_{L^\infty}$, $\|f\|_{L^1}$ and related quantities for $\partial_x f$, $\partial_x^2 f$;
3) the space $\mathcal{S}$ is an algebra; moreover it provides a good control on quantities like $\partial_x^{n_1} f^1 \partial_x^{n_2} f^2$ for $n_i \in \{0,1,2\}$ and any $f^i \in \mathcal{S}$.

1): we can decompose $f \in \mathcal{S}$ as $f = A_0 f_0^* + g$ with $\hat{g}(0) = 0$ (i.e. $A_0 = \hat{f}(0)$). Then using the property $\hat{g}(L^{-1}k) \le \|\hat{g}'\|_\infty L^{-1}k$, a simple computations shows that for all L big enough (say $L \ge 2$) it holds

$$\|R_0 g\| \le C L^{-1}\|g\|;$$

the multiplier $e^{-k^2(1-L^{-2})}$ controls the weight $1+k^4$, while $|(\hat{g}(L^{-1}\cdot))'| \le L^{-1}\|g\|$.

2): if $\|f\| < \infty$, then $\hat{f}$ decays like k^{-4} at infinity, so $\hat{f} \in L^1 \cap L^2$ and $f \in L^2 \cap L^\infty$; similarly $\hat{f}' \in L^2$ gives $x\,f(\cdot) \in L^2$ and then

$$\|f\|_{L^1} = \int |f(x)|\,\mathrm{d}x \le \left(\int (1+|x|^2)^{-1}\mathrm{d}x\right)^{1/2} (\|f\|_{L^2} + \|x\,f\|_{L^2}) \lesssim \|f\|.$$

3): if $f^i \in \mathcal{S}$, then $\widehat{\partial_x^{n_i} f^i}$ decay like $(1+k^2)^{-1}$ for any $n_i \in \{0,1,2\}$, so that $\widehat{g^i} := \widehat{\partial_x^{n_i} f^i} \in L^1$; then using $1+k^2 \lesssim 1+(k-l)^2+l^2$ we get

$$\begin{aligned}(1+k^2)|\widehat{g^1 g^2}(k)| &\lesssim \int (1+(k-l)^2)\hat{g}^1(k-l)\hat{g}^2(l)\mathrm{d}l + \int \hat{g}^1(k-l)(1+l^2)\hat{g}^2(l)\mathrm{d}l \\ &\lesssim \|(1+k^2)\hat{g}^1\|_\infty \|\hat{g}^2\|_{L^1} + \|(1+k^2)\hat{g}^2\|_\infty \|\hat{g}^1\|_{L^1} \lesssim \|f^1\|\|f^2\|.\end{aligned}$$

A similar computation shows that $\mathcal{S}$ is an algebra.

Irrelevant case: main result

Theorem (Irrelevant case)

Let $F : \mathbb{C}^3 \to \mathbb{C}$ be analytic in a neighbourhood of 0 with $d_F > 0$ and fix $\delta > 0$. Then there exists $\varepsilon > 0$ such that if $\|f\| < \varepsilon$, the equation

$$\partial_t u = \partial_x^2 u + F(u, \partial_x u, \partial_x^2 u), \quad u(1,x) = f(x)$$

has a unique solution which satisfies, for some $A = A(f, F) \in \mathbb{R}$,

$$\lim_{t\to\infty} t^{\frac{1}{2}-\delta} \| t^{\frac{1}{2}} u(t, t^{\frac{1}{2}} \cdot) - A f_0^*(\cdot) \| = 0. \tag{1}$$

- The constant A is not computed explicitly but rather the limit of a sequence $\{A_n\}_n$ constructed iteratively.
- This is where the RG approach shows its effectiveness compared to a more perturbative approach: the initial guess $A_0 = \int f(x)\,\mathrm{d}x$ is **wrong** and trying to expand u around $A_0 f_0^*$ can't give any results.
- Contrary to more classical PDE approaches, we need to restrict to small initial data, $\|f\| < \varepsilon$; however no assumption whatsoever on the sign of F.

Remarks on Theorem 1

a) Estimate (1) translates in Fourier space as

$$|\hat{u}(k,t) - Ae^{-tk^2}| \le C\, t^{\delta - \frac{1}{2}}\,(1 + t^2 k^4)^{-1} \tag{2}$$

for some constant $C > 0$, uniformly over (k,t); taking smoother initial data improves the decay on the r.h.s. accordingly.

b) The proof also works in $\mathbb{R}^N$ and for Δu replaced by a fractional Laplacian:

$$\partial_t u = -(-\Delta)^{\beta/2} + F(u)$$

where F is analytic in u and its spatial derivatives up to order β; in this case $u(t,x) \sim At^{-\frac{N}{\beta}} f^*(t^{-\frac{1}{\beta}} x)$ for f^* given by $\hat{f}^*(k) = e^{-|k|^\beta}$.

The RG map is given by $R_L f = L^N u(L^\beta t, Lx)$; for monomials

$$F = \Pi_{i,j}(\partial_j^{a_j} u)^{n_{ij}}$$

the degree d_F is defined as

$$d_F = \sum_{i,j}(N + a_j)n_{ij} - (N + \beta);$$

similarly for general d_F; the result then applies in the regime $d_F > 0$.

Proof of Theorem 1

We start by discussing local existence in $\mathcal{S}$; write the PDE in mild form

$$u_t = e^{(t-1)\partial_x^2} f + \int_0^{t-1} e^{(t-1-s)\partial_x^2} \bar{F}(u_s)\,\mathrm{d}s =: (u_f)_t + N(u)_t \tag{3}$$

where we used the shortcut notation $\bar{F}(u) = F(u, \partial_x u, \partial_x^2 u)$.

We solve (3) by a contraction argument: define

$$\|u\|_L := \sup_{t\in[1,L^2]} \|u_t\|, \quad B_f := \big\{ u \,\big|\, \|u - u_f\|_L \leq \|f\| \big\};$$

then $T(u) = u_f + N(u)$ is a contraction on B_f for $\|f\| \leq \varepsilon = \varepsilon(F, L)$ small.

To see this, expand F in its Taylor series around 0, so that

$$\widehat{\bar{F}(u)} = \sum_{\mathbf{n}\in\mathbb{N}^3} a_{\mathbf{n}} \widehat{u}^{\,*n_1} * \widehat{\partial_x u}^{\,*n_2} * \widehat{\partial_x^2 u}^{\,*n_3};$$

we can now use the properties of $\|\cdot\|$ to estimate the action of the heat semigroup on each monomial term as follows:

$$
\begin{aligned}
&\int_0^{t-1} e^{-(t-s-1)k^2} \left| \widehat{u}^{*n_1} * \widehat{\partial_x u}^{*n_2} * \widehat{\partial_x^2 u}^{*n_3} \right|(k)\,\mathrm{d}s \\
&\qquad \lesssim (C\|u\|)^{n_1+n_2+n_3}(1+|k|^2)^{-1}\int_0^{t-1} e^{-sk^2}\mathrm{d}s \\
&\qquad \leq (C\|u\|_L)^{n_1+n_2+n_3}(1+|k|^4)^{-1}.
\end{aligned}
$$

A similar estimate holds for the monomial terms in $\widehat{\bar{F}(u)}'$ as well; since F is analytic, $|a_{\mathbf{n}}| \leq (C_F)^{n_1+n_2+n_3}$. Summing over $\mathbf{n}$ we get a convergence series for $\|u\|_L$ small enough and moreover ($d_F > 0$ implies $n_1 + n_2 + n_3 \geq 2$)

$$
\|N(u)\|_L \leq C_{L,F}\|u\|_L^2.
$$

Finally, since $\|u_f\|_L \lesssim \|f\|$ it holds $\|u\|_L \lesssim \|f\|$ and so $\|T(u)\| \lesssim \|u\|_L \leq \|f\|$ once we choose $\|f\| \leq \varepsilon(F, L)$ small enough. Thus T maps B_f into itself.

A similar argument gives the estimate

$$
\|N(u_1) - N(u_2)\|_L \leq C_F L^2(\|u_1\|_L + \|u_2\|_L)\|u_1 - u_2\|_L
$$

showing that T is a contraction on B_f (for $\|f\| \leq \varepsilon(F, L)$). The unique solution u satisfies $u(L^2, \cdot) = u_f(L^2, \cdot) + \nu(\cdot)$ with $\|\nu\| \leq C_{L,F}\|f\|^2$.

To study the asymptotic behaviour of the solution, we now set up an iterative argument based on the decomposition $f = A_0 f_0^* + g_0$ with $A_0 = \hat{f}(0)$, $\hat{g}(0) = 0$.

The reason for this decomposition comes: i) from R_0 being contractive on g; ii) the fact that $\|g_0\| = \|f - \hat{f}(0) f_0^*\| \leq C\|f\|$.

We have the relation $Rf = Lu(L^2, L\,\cdot) = R_0 f + L\nu(L\,\cdot)$ and we can now decompose it again as $Rf = A_1 f_0^* + g_1$ for the choice

$$A_1 := A_0 + \hat{\nu}(0), \qquad g_1 := R_0 g_0 + L\nu(L\,\cdot) - \hat{\nu}(0) f_0^*.$$

Since $\|\nu\| \leq C_{L,F}\|f\|^2$, we have

$$|A_1 - A_0| = |\hat{\nu}(0)| \leq \|\nu\| \leq C_{L,F}\|f\|^2$$
$$\|L\nu(L\,\cdot) - \hat{\nu}(0) f_0^*\| \leq 2\|\nu\| \leq C_{L,F}\|f\|^2;$$

combined with $\|R_0 g_0\| \leq CL^{-1}\|g_0\|$ we deduce that

$$\|g_1\| \leq CL^{-1}\|g_0\| + C_{L,F}\|f\|^2 \leq (CL^{-1} + C_{L,F}\|f\|)\|f\| \leq L^{-(1-\delta)}\|f\|$$

where the last inequality holds for $\|f\| \leq \varepsilon(L,F)$ and $L = L(\delta)$ large enough, e.g.

$$2C \leq L^{\delta}, \quad 2C_{L,F}\varepsilon \leq L^{-(1-\delta)}.$$

It now remains to iterate this procedure: set $f_n := R_{L^n,F} f = A_n f_0^* + g_n$ and define the next iterative decomposition by

$$A_{n+1} = A_n + \hat{\nu}_n(0), \quad g_{n+1} = R_0 g_n + L\nu_n(L\,\cdot) - \hat{\nu}_n(0) f_0^*.$$

Assume inductively that

$$\|f_n\| \leq C\|f\|, \quad \|g_n\| \leq CL^{-(1-\delta)n}\|f\|;$$

we can then go through the same analysis as before, only observing that replacing F by F_{L^n} will now produce a factor L^{-nd_F} in front of all constants depending on F:

$$\begin{aligned} \|\nu_n\| &\leq C_{F,L} L^{-nd_F}\|f\|^2 \\ |A_{n+1} - A_n| &\leq C_{L,F} L^{-nd_F}\|f\|^2 \\ \|g_{n+1}\| &\leq CL^{-1}\|g_n\| + C_{L,F} L^{-nd_F}\|f\|^2 \leq CL^{-(1-\delta)(n+1)}\|f\|. \end{aligned}$$

The inductive assumption holds and we deduce that $\{A_n\}$ behaves like a geometric series, $A_n \to A$ with $|A - A_0| = |A - \hat{f}(0)| \leq C_{L,F}\|f\|^2$ and that

$$\|L^n u(L^{2n}, L\,\cdot) - A f_0^*\| \lesssim |A - A_n| + \|g_n\| \lesssim L^{-(1-\delta)n}\|f\|.$$

But this is exactly our claim for $t = L^{2n}$; the same technique allows to extend the bound to $t \in [L^{2n}, L^{2n+2}]$ giving the conclusion. □

Marginal cases

There are two possible marginal cases: **cubic nonlinearity**

$$\partial_t u = \partial_x^2 u - u^3 + G(u, \partial_x u, \partial_x^2 u)$$

or **Burgers nonlinearity**

$$\partial_t u = \partial_x^2 u + 2u\partial_x u + H(u, \partial_x u, \partial_x^2 u)$$

where H and G are irrelevant nonlinearities. These two cases behave quite differently; we start analysing the first one.

The sign $-u^3$ is needed for long time existence, otherwise solutions might blow up in finite time. For technical convenience, we will assume that the Taylor expansion of G starts at degree 4 or higher; otherwise there is an initial "crossover time" during which G dominates u^3.

We can then rescale $u = \lambda^{1/2}\tilde{u}$ in such a way that $\hat{f}(0) = 1$ and treat

$$\partial_t u = \partial_x^2 u - \lambda u^3 + G_\lambda(u, \partial_x u, \partial_x^2 u)$$

where $G_\lambda(z) = \lambda^{-1/2} G(\lambda^{1/2} z)$ is of order $\lambda^{3/2}$ for λ small. We can write $f = f_0^* + g$ with $\hat{g}(0) = 0$.

Marginal case I: main result

Theorem (Marginal case, cubic nonlinearity)

Let $G : \mathbb{C}^3 \to \mathbb{C}$ be analytic in a neighbourhood of 0 with $d_G > 0$. Then for any $\delta > 0$ there exist $\lambda_0, \varepsilon > 0$ such that for any $\lambda \le \lambda_0$, $\|g\| \le \varepsilon$, the solution to

$$\partial_t u = \partial_x^2 u - \lambda u^3 + G_\lambda(u, \partial_x u, \partial_x^2 u)$$

with initial data $f_0^* + g$, with $\hat{g}(0) = 0$, satisfies

$$\lim_{t\to\infty} (\log t)^{1-\delta} \left\| t^{\frac{1}{2}} u(t, t^{\frac{1}{2}} \cdot) - \left(\frac{\lambda}{2\sqrt{3}\pi} \log t \right)^{-\frac{1}{2}} f_0^*(\cdot) \right\| = 0. \tag{4}$$

- The marginal nonlinearity yields a **logarithmic correction** and a worsened rate of convergence.
- The rescaled solution $t^{\frac{1}{2}} u(t, t^{\frac{1}{2}} \cdot)$ converges to 0; although there is a nonlinearity, at leading order it behaves like a Gaussian f_0^*.
- Contrary to Theorem 1, the constant in (4) is explicit. This is because here the iterative sequence $\{A_n\}$ will actually converge to 0; the term $(2\sqrt{3}\pi)^{-1}$ comes from a higher order Picard iteration around f_0^*.

Idea of proof of Theorem 2

Let us introduce the shortcut notations $P_t = e^{t\partial_x^2}$, $\bar{G}(u) = G(u, \partial_x u, \partial_x^2 u)$ and

$$N_3(u)_t := \int_1^t P_{t-s} u_s^3 \,\mathrm{d}s, \quad N_G(u)_t := \int_1^t P_{t-s} \bar{G}(u_s) \,\mathrm{d}s.$$

Since u^3 is marginal, we need to handle its effect on u explicitly, which will lead to the logarithmic correction. For future use, we deal with initial data $f = Af_0^* + g$ with $|A| \leq 1$ (although $A_0 = 1$).

The mild formulation reads

$$u_A(t) = P_{t-1} f + N(u_A)_t = AP_{t-1} f_0^* + P_{t-1} g - \lambda N_3(u_A) + N_G(u_A);$$

existence and uniqueness for u_A can established as before; also let u_A^* solve

$$u_A^*(t) = AP_{t-1} f_0^* - \lambda N_3(u_A^*)(t)$$

One can show that, uniformly over $|A| \leq 1$, $\|g\| \leq \varepsilon$ and $\lambda \leq \lambda_0$ small, it holds

$$\|N_3(u_A) - N_3(u_A^*)\|_L \leq C_L(A^2 \|g\| + \|g\|^3) + C_{L,G} \lambda^{3/2}$$

where the factor $\lambda^{3/2}$ comes from the rescaling G_λ and our assumptions on G.

Define ν by the relation $u_A(L^2,\cdot) = P_{L^2-1} f + \nu$. We can expand u_A further as follows: set

$$\nu^* = N_3(P_{\cdot-1} f_0^*)_{L^2} = \int_1^{L^2} P_{L^2-s}(P_{s-1} f_0^*)^3 \,\mathrm{d}s$$

and define w by the relation

$$u_A(L^2,\cdot) = P_{L^2-1} f - \lambda A^3 \nu^* + w.$$

After some calculations, one obtains (by an intermediate comparison with u_A^*)

$$\|N_3(u_A^*)(L^2) - A^3\nu^*\| \leq C_L \,\lambda\, A^5$$
$$\|w\| \leq C_{L,G}\,\lambda\,(A^2\|g\| + \|g\|^3 + \lambda^{1/2} + \lambda A^5).$$

We have $R_L f = R_0 f + L\nu(L\,\cdot)$, which we can decompose as $R_L f = A_1 f_0^* + g_1$ with $\hat{g}_1(0) = 0$. But then it holds

$$A_1 = A_0 + \hat{\nu}(0) = A_0 - \lambda\beta A_0^3 + \hat{w}(0) \quad \text{for } \beta := \widehat{\nu^*}(0)$$

and $|A_1 - A_0 + \lambda\beta A_0^3| \leq |\hat{w}(0)| \leq \|w\|$ to which the previous estimate applies. Finally, as in Theorem 1, $\|g_1\| \leq C L^{-1}\|g_0\| + C_{L,G}\lambda$.

We are now ready to run the iteration argument for the sequence (A_n, g_n) related to the decomposition $R_{L^n} f = A_n f_0^* + g_n$ with $\hat{g}_n(0) = 0$.

G will produce a term L^{-nd_G}, improving some of the previous bounds, but the terms associated to the nonlinearity u^3 stay untouched. Contrary to Theorem 1, the sequence $\{A_n\}$ will converge to 0 and we need to keep track of the higher order A^5.

Repeating the previous calculations, using the decomposition $\nu_n = -\lambda A_n^3 \nu^* + w_n$ and estimating w_n, one arrives at the estimates:

$$|A_{n+1} - A_n + \lambda\beta A_n^3| \leq C_{L,G}\, \lambda\, (A_n^2 \|g_n\| + \|g_n\|^3 + \lambda^{1/2} L^{-nd_G} + \lambda A_n^5) \quad (5)$$

$$\|g_{n+1}\| \leq C L^{-1} \|g_n\| + C_{L,G} \lambda (A_n^3 + \|g_n\|^3 + \lambda^{1/2} L^{-nd_G} + \lambda A_n^5) \quad (6)$$

together with the information that A_n is decreasing to 0.

Thus in the above, the leading terms are given respectively by A_n^3 in (6) and $A_n^2 \|g_n\| + A_n^5$ in (5); A_n must satisfy

$$A_{n+1} - A_n + \lambda\beta A_n^3 = \mathcal{O}(A_n^5)$$

which implies it being at first order of the form $A_n \sim (2\lambda\beta n)^{-1/2}$.

Inserting again this information in (5),(6) gives the asymptotic behaviour

$$A_n = (2\lambda\beta n)^{-1/2} + \mathcal{O}(n^{-1}), \quad \|g_n\| \leq C_{L,G}\, n^{-3/2}.$$

By $R_{L^n} f = A_n f_0^* + g_n$ and the above we deduce that, for any $\delta > 0$,

$$n^{1-\delta} \| L^n u(L^{2n}, L\,\cdot) - (2\lambda\beta n)^{-1/2} f_0^* \| \lesssim n^{-\delta} \to 0;$$

the constant β can be computed explicitly as it was given by

$$\beta = \frac{1}{4\pi^2} \int_{\mathbb{R}} \int_1^{L^2} P_{L^2-s}(P_{s-1} f_0^*)^3(x)\, \mathrm{d}s \mathrm{d}x = \frac{\log L}{2\sqrt{3}\pi}.$$

Taking $t = L^{2n}$, so that $2\log L n = \log t$, the above estimate becomes

$$(\log t)^{1-\delta} \left\| t^{1/2}\, u(t, t^{1/2}\,\cdot) - \Big(\frac{\log t}{2\sqrt{3}\pi}\Big)^{-1/2} f_0^* \right\| \lesssim (\log t)^{-\delta} \to 0$$

which shows the claim for $t = L^{2n}$. Extending it on intervals $[L^{2n}, L^{2n+2}]$ can be done by standard arguments. □

Marginal case II: Burgers nonlinearity

Consider now the marginal case given by

$$\partial_t u = \partial_x^2 u + \partial_x(u^2) + H(u, \partial_x u, \partial_x^2 u) \tag{7}$$

with irrelevant H, namely $d_H > 0$.

To study (7), we start treating $H \equiv 0$, i.e. viscous Burgers equation. We can reduce it to the heat equation by the **Cole-Hopf transformation**: set

$$\psi(t,x) = \exp\Big(\int_{-\infty}^{x} u(t,y)\mathrm{d}y \Big),$$

then ψ solves (HE): $\partial_t \psi = \partial_x^2 \psi$.

Observe that $\psi(1,\cdot)$ is not integrable, instead obeys the **boundary conditions**

$$\lim_{x\to-\infty} \psi(1,x) = 1, \quad \lim_{x\to+\infty} \psi(1,x) = \exp\Big(\int_{\mathbb{R}} u(1,y)\mathrm{d}y \Big);$$

thus we need to modify our previous analysis for R_0 to find fixed points of the form of "**Gaussian fronts**" satisfying boundary conditions.

The scale invariance for (HE) in the presence of b.c. is $\psi_L(t,x) = \psi(L^2 t, Lx)$ ($\alpha = 0$), contrary to $u_L(t,x) = Lu(L^2 t, Lx)$ ($\alpha = 1$) as before.

Family of fixed points for RG

The asymptotic behaviour for (HE) in the presence of boundaries can be computed explictly: suppose $\psi(1,\pm\infty)=\psi_\pm$, then

$$\begin{aligned}\psi(t+1,t^{1/2}x) &= \Big(e^{t\partial_x^2}\psi(1,\cdot)\Big)(t^{1/2}x)\\ &= (4\pi)^{-1/2}\int e^{-\frac{(x-y)^2}{4}}\psi(1,t^{1/2}y)\,\mathrm{d}y\\ &\xrightarrow[t\to\infty]{} (4\pi)^{-1/2}\int e^{-\frac{(x-y)^2}{4}}(\psi_-\mathbb{1}_{(-\infty,0)}(y)+\psi_+\mathbb{1}_{[0,+\infty)}(y))\,\mathrm{d}y\end{aligned}$$

Since $\psi_-=1$, setting $A=\psi_+-1$ we can write the last expression as

$$\psi_A(x)=1+Ae(x),\quad e(x)=(4\pi)^{-1/2}\int_{-\infty}^x e^{-y^2/4}\,\mathrm{d}y.$$

The family $\{\psi_A^*\}_{A\in\mathbb{R}}$ gives a 1-parameter family of fixed points for the RG associated to (HE) and scaling $\psi_L(t,x)=\psi(L^2t,Lx)$ (equiv. $\alpha=0$).

To see this, observe that $\psi_A^*=e^{\partial_x^2}h_A$ for h_A satisfying $h_A(Lx)=h_A(x)$; then

$$R_L^{\alpha=0}\psi_A^*=\big(e^{L^2\partial_x^2}h_A\big)(L\,\cdot)=e^{\partial_x^2}\big(h_A(L\,\cdot)\big)=\psi_A^*.$$

Role of different scalings

Going back to Burgers via the inverse transform $u = (\log \psi)'$, we find a 1-parameter family of fixed points for RG (for $\alpha = 1$) given by

$$f_A^*(x) = \partial_x(\log \psi_A^*)(x) = \partial_x(\log(1 + Ae(x))) = \frac{Ae'(x)}{1 + Ae(x)}.$$

The parameter A can be recovered from f_A^* by means of the relation

$$\log(1 + A) = \int_{\mathbb{R}} f_A^*(x)\,\mathrm{d}x.$$

The relation $u = \log(\psi)'$ explains the correspondence of the different scalings $\alpha = 0$ and $\alpha = 1$ for Burgers and (HE) (equiv. $\{\psi_A^*\}_A$ and $\{f_A^*\}_A$):

$$\begin{aligned} R_L^{\alpha=1} f &= Lu(L^2, L\,\cdot) = L\frac{\partial_x \psi(L^2, L\,\cdot)}{\psi(L^2, L\,\cdot)} \\ &= \partial_x(\log \psi(L^2, L\,\cdot)) = \partial_x(\log R_L^{\alpha=0}\psi_1). \end{aligned}$$

Marginal case II: main result

Theorem (Marginal case, Burgers nonlinearity)

Let $H : \mathbb{C}^3 \to \mathbb{C}$ be analytic in a neighbourhood of 0 with $d_H > 0$ and fix $\delta > 0$. Then there exists $\varepsilon > 0$ such that if $\|f\| < \varepsilon$, the equation

$$\partial_t u = \partial_x^2 u + \partial_x(u^2) + H(u, \partial_x u, \partial_x^2 u), \quad u(1,x) = f(x)$$

has a unique solution which satisfies, for some $A = A(f, F) \in \mathbb{R}$,

$$\lim_{t\to\infty} t^{\frac{1}{2}-\delta} \| t^{\frac{1}{2}} u(t, t^{\frac{1}{2}} \cdot) - f_A^*(\cdot)\| = 0. \tag{8}$$

- Like in Theorem 1, the constant A cannot be explicitly computed but is given as the limit of an iterative sequence $\{A_n\}_n$.
- Differently from Theorem 2, in this marginal case there are **no logarithmic corrections**, same rate of as in Theorem 1.
- The rescaled solution u is **not attracted by the Gaussian family** $\{Af_0^*\}_A$ of fixed points to (HE) but instead by $\{f_A^*\}_A$ fixed points for Burgers.

Idea of proof

The proof is similar to that of Theorem 1, up to the change of fixed point. Decompose $f = f^*_{A_0} + g_0$ for $\hat{g}_0(0) = 0$, which determines A_0 by the relation

$$\log(1 + A_0) = \int f(x)\,\mathrm{d}x.$$

Observe that $\|f\|_{L^1} \le C\|f\| \le C\varepsilon$, implying $|A_0| \le C\varepsilon$. One can then show that

$$\|g_0\| \le C\|f\|$$

using the explicit expression for f^*_A and the bound on $|A_0|$.

Local existence is proved as before; to study R_L, write the solution as

$$u(t,x) = u^*_{A_0}(t,x) + \nu(t,x), \quad u^*_{A_0}(t,x) := t^{-1/2}\, f^*_{A_0}(t^{-1/2}\, x)$$

so that ν satisfies ($u^*_{A_0}$ solution for $H \equiv 0$ with data $f^*_{A_0}$))

$$\partial_t \nu = \partial_x^2 \nu + \partial_x\left[(u^*_{A_0} + \nu)^2 - (u^*_{A_0})^2\right] + \bar{H}(u), \quad \nu(1,x) = g_0$$

with the shortcut notation $\bar{H}(u) = H(u, \partial_x u, \partial_x^2 u)$.

We have

$$R_L f(\cdot) = Lu(L^2, L\,\cdot) = f^*_{A_0}(\cdot) + L\nu(L^2, L\,\cdot)$$

which we can decompose again as $R_L f = f^*_{A_1} + g_1$ for $\hat{g}_1(0) = 0$ and

$$\log(1 + A_1) = \int [f^*_{A_0} + L\nu(L^2, Lx)]\,\mathrm{d}x = \log(1 + A_0) + \int \nu(L^2, x)\,\mathrm{d}x.$$

It follows from the PDE for ν that

$$\left|\frac{\mathrm{d}}{\mathrm{d}t} \int \nu(t, x)\,\mathrm{d}x\right| \leq \int |\bar{H}(u)(x)|\,\mathrm{d}x \leq \|\bar{H}(u)\|;$$

the term $\bar{H}(u)$ can be controlled as in Theorem 1, yielding (for $|A_i|$ small)

$$|A_1 - A_0| \leq C\left|\int \nu(L^2, x)\,\mathrm{d}x\right| \leq C_{L,H}\|f\|^2.$$

By the definition of g_1, the above estimates and the contractivity of $\|\cdot\|$, we have

$$\|g_1\| \leq CL^{-1}\|g_0\| + C_{L,H}\|f\|^2$$

To see this, use the PDE ν in mild form: $L\nu(L^2, L\,\cdot) = R_0 g_0 + h_0$ for a remainder term h_0 which is controlled by $\|u\|_L \leq C_{L,H}\|f\|^2$.

Choosing suitable $L = L(\delta)$ and $\varepsilon = \varepsilon(L, H)$, using $\|g_0\| \leq C\|f\|$, we then have

$$\|g_1\| \leq L^{-(1-\delta)}\|f\|.$$

It only remains to run the iterative argument. Since H_{L^n} decays with L^{-nd_H}, adding it in front $C_{L,H}$ in the previous estimates yields

$$|A_{n+1} - A_n| \leq C_{L,H}\, L^{-nd_H}\|f\|^2,$$
$$\|g_n\| \leq CL^{-(1-\delta)n}\|f\|.$$

The rest of the argument is exactly as in the proof of Theorem 1:

- $A_n \to A$ geometrically;
- $R_{L^n} f = f^*_{A_n} + g_n$ then gives an estimate for $\|R_{L^n} f - f^*_A\|$
- this proves the statement for $t = L^{2n}$, then extend to general t. □

Final discussion: comparison with known results

We conclude with an heuristic discussion on the results presented and the related literature for the **heat equation with absorption**:

$$\partial_t u = \partial_x^2 u - u^p, \quad p > 1. \tag{9}$$

The PDE (9) is invariant under $u_L(t,x) = L^\alpha u(L^2 t, Lx)$ for $\alpha = 2/(p-1)$. To understand what happens in the case $p < 3$, one first looks for self-similar solutions $u(t,x) = t^{-\frac{1}{p-1}} f^*(t^{\frac{1}{2}} x)$, corresponding to the ODE

$$f'' + \frac{1}{2} x f' + \frac{f}{p-1} - f^p = 0.$$

The following results are known for this ODE:

1) for $p \in (1,3)$, there exists a positive solution f_1^* with almost Gaussian decay;
2) for any $p \in (1,\infty)$, there exists a solution f_2^* which decays at infinity like $|x|^{-\frac{2}{p-1}}$; observe that for $p > 3$, f_2^* is heavy-tailed and not integrable.

Comparison with known results (continued)

The following are known regarding the basin of attraction of f_i^*:

1) For $p \geq 3$, for any integrable, non-negative initial data f, the asymptotic behaviour is governed by f_0^*, as in Theorems 1 and 2. This is a **global** result; the one presented here only holds for small data f and integer p, but does not require non-negativity and holds for general F (also $+u^p$ in place of $-u^p$).
2) For $p \in (1,3)$, non-negative u with suitable Gaussian decay, the asymptotic behaviour is governed by the non-trivial fixed point f_1^*.
3) If one starts with non-negative u decaying like $|x|^{-\alpha}$, $\alpha \in (0,1)$, then for $p \geq 3$ the relevant "Gaussian" fixed point becomes f_α^* for

$$\widehat{f_\alpha^*}(k) = |k|^{\alpha-1} e^{-k^2}$$

which is a fixed point of R_0 (RG for (HE)) with the right decay at infinity.

In the last case, the nonlinearity u^p is irrelevant (resp. marginal, relevant) if $p > 1 + 2/\alpha$ (resp. $=$, $<$). The dynamics is then governed respectively by f_α^*, f_2^* or the constant in space solution to $\partial_t u = -u^p$, which can be seen as a (degenerate) new fixed point.

Analogy with the theory of critical phenomena

To develop the analogy, consider an Ising model or a ϕ^4 theory on the N-dimensional lattice, at the critical point.

- $N > 4$ corresponds to the irrelevant case $p > 3$: the behaviour at the critical point is governed by the Gaussian fixed point, which may be seen as a triviality result.
- $N = 4$ becomes marginal and the Gaussian behaviour is modified by logarithmic corrections, as in Theorem 2. In ϕ^4 like in $p = 3$ here this happens because the marginal term becomes irrelevant when higher order corrections are included (in Theorem 2 we had $A_n \to 0$). This higher order irrelevance however depends on the sign of the perturbation $-u^3$.
- Other marginal nonlinearities do not have the same behaviour: think of Burgers and Theorem 2, or the nontrivial fixed point f_2^* discussed above for $p = 1 + 2/\alpha$.
- For $N < 4$ one expects the critical behaviour to be governed by a non-trivial fixed point, whose existence is much harder to establish than here.