S4F3 - Graduate Seminar on Applied Probability – WS2021

Nonperturbative Renormalization

Main reference

- Alessandro Giuliani, Vieri Mastropietro, and Slava Rychkov, `Gentle Introduction to Rigorous Renormalization Group: A Worked Fermionic Example', *ArXiv:2008.04361 [Cond-Mat, Physics:Hep-Th, Physics:Math-Ph]*, 2 September 2020, http://arxiv.org/abs/2008.04361.

- Vieri Mastropietro, *Non-Perturbative Renormalization* (Hackensack, NJ: World Scientific Publishing Co Pte Ltd, 2008).

General overview on RG

- Giovanni Jona-Lasinio, `Renormalization Group and Probability Theory', *Physics Reports* 352, no. 4–6 (October 2001): 439–58, https://doi.org/10.1016/S0370-1573(01)00042-4.

- Kenneth G. Wilson, `The Renormalization Group and Critical Phenomena', *Reviews of Modern Physics* 55, no. 3 (1983): 583–600, https://doi.org/10.1103/RevModPhys.55.583.

- P.K.Mitter:The Exact Renormalization Group, Encyclopedia in Mathematical Physics, Elsevier 2006, http://arXiv:math-ph/0505008

Complementary material (for the curious)

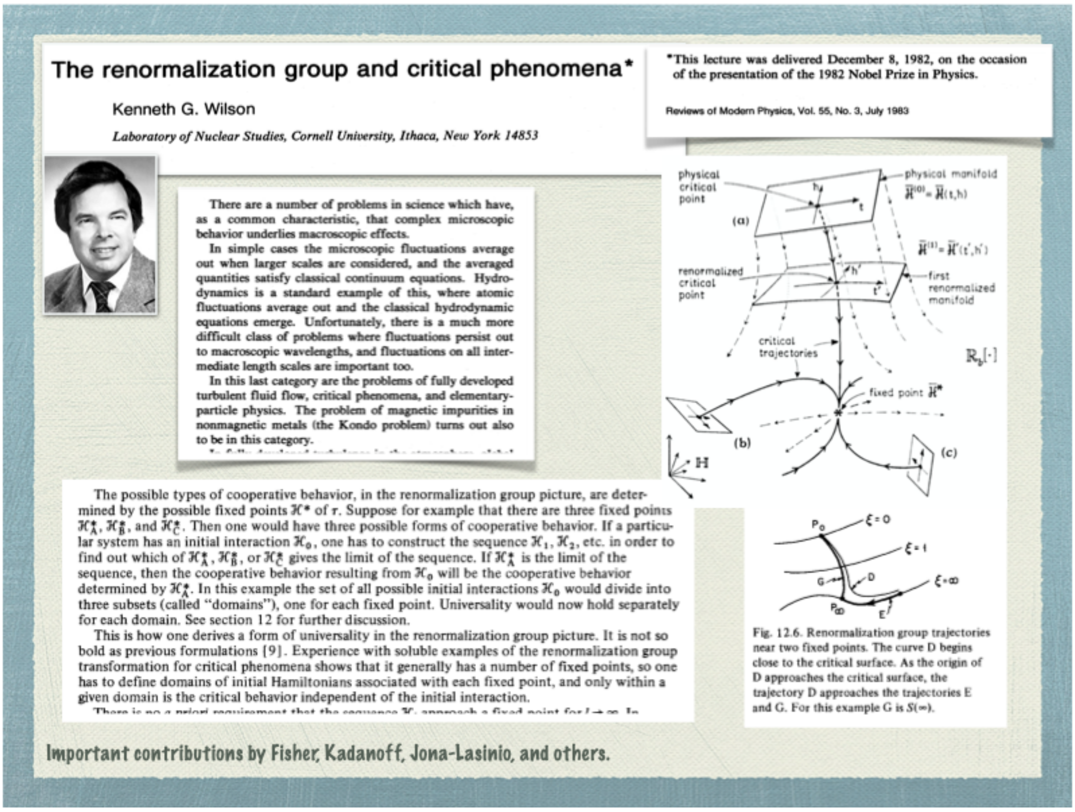
- Bertrand Delamotte, `An Introduction to the Nonperturbative Renormalization Group', *ArXiv:Cond-Mat/0702365*, 15 February 2007, http://arxiv.org/abs/cond-mat/0702365.

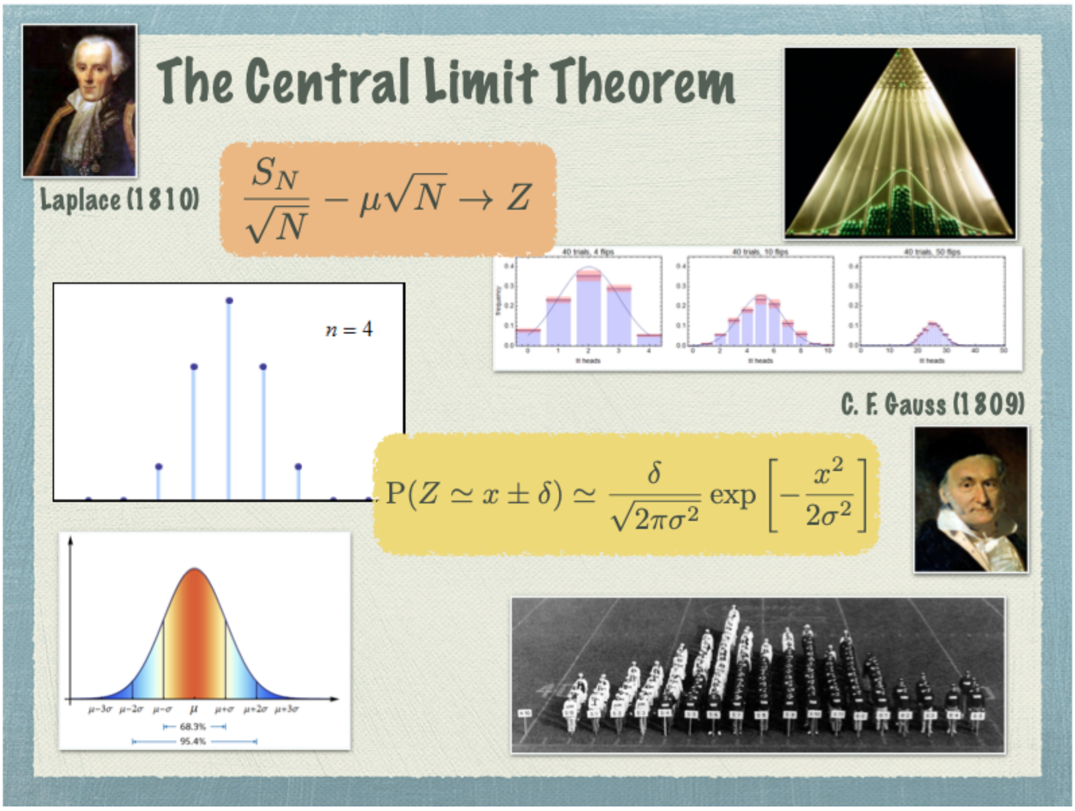
- Giovanni Gallavotti, `Renormalization Theory and Ultraviolet Stability for Scalar Fields via Renormalization Group Methods', *Reviews of Modern Physics* 57, no. 2 (1 April 1985): 471–562, https://doi.org/10.1103/RevModPhys.57.471.

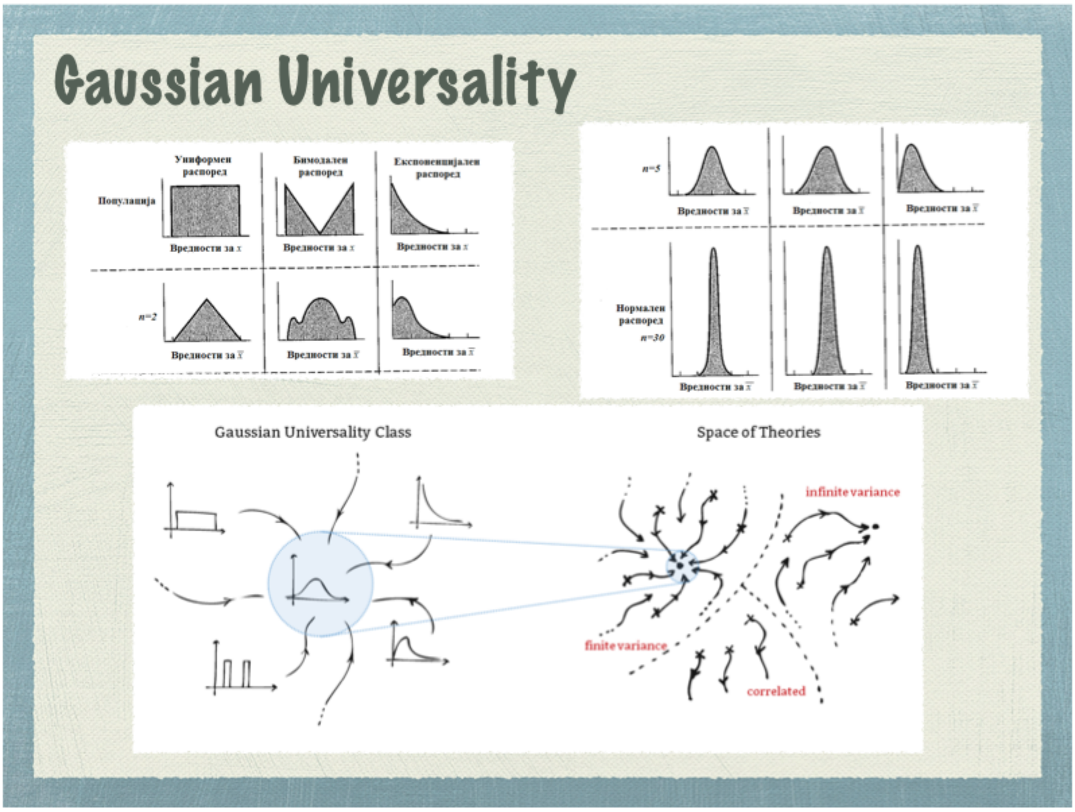
- Manfred Salmhofer, *Renormalization: An Introduction*, 1st Corrected ed. 1999, Corr. 2nd printing 2007 edition (Berlin; New York: Springer, 2007).

- Joseph Polchinski, `Renormalization and Effective Lagrangians', *Nuclear Physics B* 231, no. 2 (January 1984): 269–95, https://doi.org/10.1016/0550-3213(84)90287-6.

- David C. Brydges, Roberto Fernández, *Functional Integrals and Their Applications*, 1993.


The renormalization group and critical phenomena*
Kenneth G. Wilson
Laboratory of Nuclear Studies, Cornell University, Ithaca, New York 14853
*This lecture was delivered December 8, 1982, on the occasion of the presentation of the 1982 Nobel Prize in Physics.
Reviews of Modern Physics, Vol. 55, No. 3, July 1983
There are a number of problems in science which have, as a common characteristic, that complex microscopic behavior underlies macroscopic effects.
In simple cases the microscopic fluctuations average out when larger scales are considered, and the averaged quantities satisfy classical continuum equations. Hydrodynamics is a standard example of this, where atomic fluctuations average out and the classical hydrodynamic equations emerge. Unfortunately, there is a much more difficult class of problems where fluctuations persist out to macroscopic wavelengths, and fluctuations on all intermediate length scales are important too.
In this last category are the problems of fully developed turbulent fluid flow, critical phenomena, and elementary-particle physics. The problem of magnetic impurities in nonmagnetic metals (the Kondo problem) turns out also to be in this category.
The possible types of cooperative behavior, in the renormalization group picture, are determined by the possible fixed points $\mathcal{H}^*$ of τ. Suppose for example that there are three fixed points $\mathcal{H}^*_A$, $\mathcal{H}^*_B$, and $\mathcal{H}^*_C$. Then one would have three possible forms of cooperative behavior. If a particular system has an initial interaction $\mathcal{H}_0$, one has to construct the sequence $\mathcal{H}_1$, $\mathcal{H}_2$, etc. in order to find out which of $\mathcal{H}^*_A$, $\mathcal{H}^*_B$, or $\mathcal{H}^*_C$ gives the limit of the sequence. If $\mathcal{H}^*_A$ is the limit of the sequence, then the cooperative behavior resulting from $\mathcal{H}_0$ will be the cooperative behavior determined by $\mathcal{H}^*_A$. In this example the set of all possible initial interactions $\mathcal{H}_0$ would divide into three subsets (called "domains"), one for each fixed point. Universality would now hold separately for each domain. See section 12 for further discussion.
This is how one derives a form of universality in the renormalization group picture. It is not so bold as previous formulations [9]. Experience with soluble examples of the renormalization group transformation for critical phenomena shows that it generally has a number of fixed points, so one has to define domains of initial Hamiltonians associated with each fixed point, and only within a given domain is the critical behavior independent of the initial interaction.
physical critical point
physical manifold
(a)
renormalized critical point
first renormalized manifold
critical trajectories
fixed point
(b)
(c)
H
$\xi = 0$
$\xi = 1$
$\xi = \infty$
Fig. 12.6. Renormalization group trajectories near two fixed points. The curve D begins close to the critical surface. As the origin of D approaches the critical surface, the trajectory D approaches the trajectories E and G. For this example G is $S(\infty)$.
Important contributions by Fisher, Kadanoff, Jona-Lasinio, and others.


The Central Limit Theorem
Laplace (1810)
$\frac{S_N}{\sqrt{N}} - \mu\sqrt{N} \to Z$
$n = 4$
C. F. Gauss (1809)
$P(Z \simeq x \pm \delta) \simeq \frac{\delta}{\sqrt{2\pi\sigma^2}} \exp\left[-\frac{x^2}{2\sigma^2}\right]$
68.3%
95.4%



CLT via RG

CLT: $(X_n)_{n\geqslant 1}$ iid, $\mathrm{Var}(X_k) < \infty$, $\mathbb{E}[X_k] = 0$. $n \to \infty$

$$Z_n = \frac{X_1 + \cdots X_{2^n/2} + X_{2^n/2+1} + \cdots + X_{2^n}}{2^{n/2}} = \frac{Z_{n-1} + \tilde{Z}_{n-1}}{2^{1/2}}, \qquad H_n = \frac{Z_{n-1} - \tilde{Z}_{n-1}}{2^{1/2}}$$

$$Z_{n-1} = \frac{Z_n + H_n}{2^{1/2}}, \quad \tilde{Z}_{n-1} = \frac{Z_n - H_n}{2^{1/2}},$$

Dynamics in the space of distributions. $Z_n \sim p_n(z)\mathrm{d}z$. RG map $\mathscr{R}$

$$p_n(z) = 2^{1/2} \int p_{n-1}(2^{1/2}z - z')\,p_{n-1}(z')\,\mathrm{d}z' = 2^{1/2}(p_{n-1} * p_{n-1})(2^{1/2}z) = \mathscr{R}(p_{n-1})(z)$$

Let

$$\gamma_\sigma(z) = \frac{e^{-z^2/2\sigma^2}}{(2\pi\sigma^2)^{1/2}}$$

then

$$\mathscr{R}\gamma_\sigma = \gamma_\sigma$$

for any $\sigma > 0$. The variance is an invariant of $\mathscr{R}$.

Perturbation $p(z) = \gamma(z)(1+h(z))$

$$\mathscr{R}(\gamma(1+h))(z) = \gamma(z) + 2^{1/2}\int \gamma(2^{1/2}z - z')\gamma(z')2h(z')\mathrm{d}z' + O(h^2)$$

$$= \gamma(z) + \gamma(z)2\int \gamma(t)h(2^{-1/2}z + 2^{-1/2}t)\mathrm{d}t + O(h^2)$$

$$h_1(z) = z \Rightarrow \mathscr{R}(\gamma(1+\varepsilon h_1)) = \gamma(1+\varepsilon 2^{1/2}h_1) + O(\varepsilon^2)$$

$$h_2(z) = z^2 - 1 \Rightarrow \mathscr{R}(\gamma(1+\varepsilon h_2)) = \gamma(1+\varepsilon h_2) + O(\varepsilon^2)$$

$$h_n(z) = \cdots \Rightarrow \mathscr{R}(\gamma(1+\varepsilon h_2)) = \gamma(1+\varepsilon h_2) + O(\varepsilon^2)$$

$$h_n(2^{-1/2}z + 2^{-1/2}t) = (2^{-1/2}z)^k h_{n-k}(2^{-1/2}t)$$

Organization of the seminar.

$$e^{-H'(\psi)} := \int e^{-H(\varphi + t_\gamma(\psi))}\mu(\mathrm{d}\varphi)$$

$$H' = \mathscr{R}_\gamma(H, \mu)$$

$\gamma > 1$ rescaling factor. H, H' are effective actions at different scales μ describes the fluctuations.

$$\mathscr{R}_\gamma \mathscr{R}_{\gamma'} = \mathscr{R}_{\gamma\gamma'}$$

- Definition of the model, Berezin integrals, definition of the renormalization step and the integrating out map [Eq. (5.2) / Appendix B] **Sebastian**
- Various representation for the fermionic expectations (Appendix D / Book) **Max**
- Finite volume representation and infinite volume limit (Appendix H) [facultative] **Chunqiu**
- Renomalization map in the trimmed representation and fixed point equation (Sect 5.4, 5.5, Appendix C). Introduce the Banach space $\mathscr{B}$ for effective actions. **Margherita**
- Norm bounds (Sect 5.6 / Appendix E) (needs Appendix D). Control of $\mathscr{R}_\gamma : \mathscr{B} \to \mathscr{B}$ **Francesco**
- Construction of the fixed point (Section 6) **Luca**
- Proof of the key lemma (Section 7) **Mattia**
- Fixed point via tree expansion / flow of effective couplings (Appendix J) [facultative]
- other???