Bounds on the RG operators $R_\ell^{\ell_1,\ldots,\ell_n}$

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First we recall some definitions involved in the bounds described before.

The indexes $\ell, \ell_1, \ldots, \ell_n$ varies in the set $\mathrm{TL} = \{2L, 2R, 4L, 4R, 6\mathrm{SL}, 6R, 8, 10, \cdots\}$.

The absolute value $|\ell|$ of the index ℓ is a integer $|2L| = |2R| = 2$, $|4L| = |4R| = 4$, $\cdots$.

If $\ell \in \mathrm{TL}$, we denote by B_ℓ the following Banach space:

when $\ell = 2L, 4L$ then $B_\ell = \mathbb{R}$ (with the absolute value norm),

when $\ell \neq 2L, 4L$ then B_ℓ is a Banach subspace of

$$B_\ell \subset [L^1_w(\mathbb{R}^{d|\ell|}, \mathbb{R})]^{n_\ell}$$

where $n_\ell \in \mathbb{N}$ is a suitable integer depending on $N \in \mathbb{N}$ (the number of fermionic fields) and ℓ.
The weight function $w\colon \mathbb{R}^{d|\ell|} \to \mathbb{R}$ is defined as

$$w(x) = \exp\left(\bar{C}\left(\frac{\mathrm{St}(x)}{\gamma}\right)^{\sigma}\right)$$

where $\mathrm{St}(x)$ is the Steiner diameter of the set $x = (x_1, \ldots, x_{|\ell|}) \in \mathbb{R}^{d|\ell|}$, $0 < \sigma < 1$ and $\bar{C} > 0$.

When $\ell \notin \{2L, 2R, 4L, 4R, 6\mathrm{SL}\}$, the elements $H_\ell \in B_\ell$ are indexed by objects of the form $\mathbb{A} = ((\mu_1^{\mathbb{A}}, a_1^{\mathbb{A}}), \ldots, (\mu_{|\ell|}^{\mathbb{A}}, a_{|\ell|}^{\mathbb{A}}))$, where $\mu \in \{0, \ldots, d\}$ $a \in \{1, \ldots, N\}$. The function H_ℓ is associated with the ℓ homogeneous polynomial in Ψ

$$\Psi(H_\ell) = \sum_{\mathbb{A}} \int H_\ell(x_{\mathbb{A}}, \mathbb{A}) \partial_{\mu_1^{\mathbb{A}}} \Psi_{a_1^{\mathbb{A}}}(x_1) \cdots \partial_{\mu_{|\ell|}^{\mathbb{A}}} \Psi_{a_{|\ell|}^{\mathbb{A}}}(x_{|\ell|}) \mathrm{d}x_{\mathbb{A}}.$$

The norm of H_ℓ is given by

$$\|H_\ell\|_w = \sup_{\mathbb{A}} \|H_\ell(\cdot, \mathbb{A})_{x_1=0}\|_{L_w^1(\mathbb{R}^{d|\ell|}, \mathbb{R})} = \sup_{\mathbb{A}} \int_{\mathbb{R}^{d(|\ell|-1)}} |H_\ell(x_{\mathbb{A}}, \mathbb{A})_{x_1=0}| w(x_{\mathbb{A}})_{x_1=0} \mathrm{d}(x_{\mathbb{A}} \setminus x_1).$$

When $\ell = 2L, 4L$, and thus $B_\ell = \mathbb{R}$;

$$\Psi(\nu) = \nu \sum_{a,b} \Omega_{a,b} \int \Psi_a(x) \Psi_b(x), \quad \Psi(\lambda) = \lambda \sum_{a,b,c,h} \Omega_{c,h} \Omega_{a,b} \int \Psi_a(x) \Psi_b(x) \Psi_c(x) \Psi_h(x).$$

In the case $\ell = 2R, 4R, 6\mathrm{SL}$ we have that $B_\ell \subset L_w^1(\mathbb{R}^{d|\ell|}, \mathbb{R})$

$$\Psi(H_{2R}) = \int H_{2R}(x_{\mathbb{A}}, \mathbb{A}) \partial_\mu \Psi_a \partial_{\mu'} \Psi_b \mathrm{d}x_{\mathbb{A}}, \quad \Psi(H_{4R}) = \int H_{4R}(x_{\mathbb{A}}, \mathbb{A}) \partial_\mu \Psi_a \cdots \mathrm{d}x_{\mathbb{A}}.$$

where $\mu, \mu' \neq 0$. We denote by $B_{|\ell|} = \bigoplus_{\ell', |\ell'| = |\ell|} B_{\ell'}$.

In this seminar we want to prove some analytical estimates on the RG maps

$$R_\ell^{\ell_1,\ldots,\ell_n}\colon B_{\ell_1}\times\cdots\times B_{\ell_n}\to B_\ell.$$

More precisely, for any $\ell\in\mathrm{TM}$ and $H_\ell\in B_\ell$, we have that

$$\|R_\ell^\ell(H_\ell)\|_w\leqslant\begin{cases}\gamma^{-D_2-2}\|H_{2R}\|_w \text{ for } \ell=2R\\ \gamma^{-D_4-1}\|H_{4R}\|_w \text{ for } \ell=4R\\ \gamma^{-D_{|\ell|}}\|H_\ell\|_w \qquad \text{for } \ell\geqslant 6\end{cases} \tag{1}$$

while

$$R_{2L}^{2L}(\nu)=\gamma^{-D_2}\nu, \qquad R_{4L}^{4L}(\lambda)=\gamma^{-D_4}\lambda. \tag{2}$$

In all the other cases, namely when $(n,(\ell_1,\cdots,\ell_n))\neq(1,\ell)$ and for any $h_i\in B_{\ell_i}$, we have

$$\|R_\ell^{\ell_1,\ldots,\ell_n}(h_1,\cdots,h_n)\|_w\leqslant\gamma^{-D_{|\ell|}}\rho_{|\ell|}(h_1,\cdots,h_n) \tag{3}$$

with

$$\rho_{|\ell|}(h_1,\cdots,h_n)=\begin{cases}C_\gamma^{n-1}\prod_{i=1}^n C_0^{|\ell_i|}\|h_i\|_w \text{ if } \sum_i|\ell_i|\geqslant|\ell|+2(n-1)\\ 0 \qquad\qquad \text{otherwise}\end{cases}.$$

We introduce the dilation operator $D\colon B_\ell \to B_\ell$ defined as

$$D(H_{\ell,p})(x) = \gamma^{-D_{|\ell|}-p}\gamma^{d(|\ell|-1)}H_{\ell,p}(\gamma x)$$

where

$$D_k = k[\psi] - d = k\left(\frac{d}{4} - \frac{\varepsilon}{2}\right) - d$$

p is the number of derivatives in the kernel $H_{\ell,p}$. In the particular case $\ell = 2L, 4L, 6\mathrm{SL}$ we get

$$\nu \mapsto \gamma^{\frac{d}{2}+\varepsilon}\nu, \quad \lambda \mapsto \gamma^{2\varepsilon}\lambda, \quad \mathfrak{X}(x) \mapsto \gamma^{-D_6}\gamma^{d}\mathfrak{X}(\gamma x).$$

With this definition of the operator D we define, for any $\ell \in \mathrm{TL}$,

$$R_\ell^\ell(H_\ell) = D(H_\ell)$$

In the case where $(n, \ell_1, \ldots, \ell_n) \neq (1, \ell)$ the definition of $R_\ell^{\ell_1, \ldots, \ell_n}$ is more complex.

More precisely we have the following expressions

$$R_\ell^{\ell_1, \ldots, \ell_n} = D \begin{cases} S_{|\ell|}^{\ell_1, \ldots, \ell_n} & \text{if } |\ell| \geqslant 8 \\ T_\ell^{|\ell|} S_{|\ell|}^{\ell_1, \ldots, \ell_n} & \text{if } \ell \in \{2L, 2R, 4L, 4R\} \end{cases}$$

$$R_{6\mathrm{SL}}^{\ell_1, \ldots, \ell_n} = D \begin{cases} S_6^{4L, 4L} & \text{if } (n, \ell_1, \ldots, \ell_n) = (2, 4L, 4L) \\ 0 & \text{otherwise} \end{cases}$$

$$R_{6\mathrm{R}}^{\ell_1, \ldots, \ell_n} = D \begin{cases} S_6^{\ell_1, \ldots, \ell_n} & \text{if } (n, \ell_1, \ldots, \ell_n) \neq (1, 6\mathrm{SL})(2, 4L, 4L) \\ 0 & \text{otherwise} \end{cases} .$$

where $S_{|\ell|}^{\ell_1, \ldots, \ell_n}$ is the integrating-out operator and $T_\ell^{|\ell|}$ is the trimming operator whose precise expressions will be recalled later.

Proof of inequalities (1) and (2)

We recall that in the case $\ell \neq 2L, 4L$, $(n, \ell_1, \ldots, \ell_n) = (1, \ell)$ we have that the action of the RG map is given simply given by the dilation D. More precisely when $\ell = 2R$ we have

$$R_{2R}^{2R}(H_{2R}) = \{\gamma^{-D_2}\gamma^d H_{2R,0}(\gamma x), \gamma^{-D_2-1}\gamma^d H_{2R,1}(\gamma x), \gamma^{-D_2-2}\gamma^d H_{2R,2}(\gamma x)\},$$

by the particular structure of the space B_ℓ, $H_{2R,0} = H_{2R,1} = 0$, and so

$$\|R_{2R}^{2R}(H_{2R})\|_w = \gamma^{-D_2-2}\|H_{2R}\|_{w(\cdot/\gamma)} \leqslant \gamma^{-D_2-2}\|H_{2R}\|_w.$$

For R_{4R}^{4R} the reasoning is similar and use the fact that

$$R_{4R}^{4R}(H_{4R}) = \{\gamma^{-D_4}\gamma^{3d} H_{4R,0}(\gamma x), \gamma^{-D_4-p}\gamma^{3d} H_{4R,p}(\gamma x)\}.$$

With this we have proved inequality (1). For $\ell = 2L, 4L$ we have the explicit expression of the map R_ℓ^ℓ indeed

$$R_{2L}^{2L}(\nu) = \gamma^{\frac{d}{2}+\varepsilon}\nu = \gamma^{-D_2}\nu, \quad R_{4L}^{4L}(\lambda) = \gamma^{2\varepsilon}\lambda = \gamma^{-D_4}\lambda.$$

This proves inequality (2).

$$R_\ell^\ell(H_\ell) = \{\gamma^{-D_{|\ell|}}\gamma^d H_{\ell,0}(\gamma x), \gamma^{-D_2-1}\gamma^d H_{\ell,1}(\gamma x), \gamma^{-D_2-2}\gamma^d H_{2R,2}(\gamma x), \ldots\}$$

In order to proof inequality (3) we have to give some estimates on the integration out operators $S_{|\ell|}^{\ell_1,\cdots,\ell_n}$ and on the trimming operators $T_{\ell}^{|\ell|}$. More precisely we want to prove the following theorem.

Theorem 1. *For any*

$$\|S_{|\ell|}^{\ell_1,\cdots,\ell_n}(h_1,\cdots,h_n)\|_w \leqslant \rho_{|\ell|}(h_1,\cdots,h_n) \tag{4}$$

where

$$\rho_{|\ell|}(h_1,\cdots,h_n) = \begin{cases} C_\gamma^{n-1}\prod_{i=1}^n C_0^{|\ell_i|}\|h_i\|_w \; \textit{if} \; \sum_i |\ell_i| \geqslant |\ell| + 2(n-1) \\ 0 \qquad\qquad \textit{otherwise} \end{cases}.$$

and

$$\begin{aligned} \|T_{4R}^{4,0}(H)\|_{w(\cdot/\gamma)} &\leqslant C_R\gamma\|H_{4,0}\|_w \\ \|T_{2R}^{2,1}(H)\|_{w(\cdot/\gamma)} &\leqslant C_R\gamma\|H_{2,1}\|_w \\ \|T_{2R}^{2,0}(H)\|_{w(\cdot/\gamma)} &\leqslant C_R\gamma^2\|H_{2,0}\|_w \\ \|T_{2L}^{2,0}(H)\|_{w(\cdot/\gamma)} &\leqslant C_R\|H_{2,0}\|_w \\ \|T_{4L}^{4,0}(H)\|_{w(\cdot/\gamma)} &\leqslant C_R\|H_{4,0}\|_w \end{aligned} \tag{5}$$

In this part we want to prove inequality (4) on

$$S_{|\ell|}^{\ell_1,...,\ell_n} \colon B_{\ell_1} \times \ldots \times B_{\ell_n} \to B_{|\ell|}$$

. First we recall the definition of $S_{|\ell|}^{\ell_1,...,\ell_n}$. Consider $h_1, \ldots, h_n \in B_{\ell_1} \times \cdots \times B_{\ell_n}$ and let $\tilde{H}_{|\ell|} \in B_{|\ell|}$ be defined as

$$\tilde{H}_{|\ell|} = S_{|\ell|}^{\ell_1,...,\ell_n}(h_1, \ldots, h_n)$$

then we have $\mathbb{B} = ((\mu_1, a_1), \ldots, (\mu_{|\ell|}, a_{|\ell|}))$

$$\tilde{H}_{|\ell|}(x_{\mathbb{B}}, \mathbb{B}) = \frac{1}{n!} \sum_{\substack{\mathbb{B}_1,...,\mathbb{B}_n \\ \cup_i \mathbb{B}_i = \mathbb{B}}} \sum_{\substack{\mathbb{A}_1,...,\mathbb{A}_n \\ \mathbb{B}_i \subset \mathbb{A}_i, |\mathbb{A}_i| = |\ell_i|}} (-1)^{\#} K_{((\mathbb{B}_i,\mathbb{A}_i))_{i=1,...,n}}(x_{\mathbb{B}}) \tag{6}$$

where

$$K_{((\mathbb{B}_i,\mathbb{A}_i))_{i=1,...,n}}(x_{\mathbb{B}}) = \int \mathcal{C}(x_{\bar{\mathbb{B}}}) \prod_{i=1}^{n} h_i(x_{\mathbb{A}_i}, \mathbb{A}_i) \mathrm{d}x_{\bar{\mathbb{B}}}$$

where

$$\bar{\mathbb{B}} = \left(\bigcup_{i=1,\ldots,n} \mathbb{A}_i \right) \setminus \mathbb{B}$$

and

$$\mathcal{C}(x_{\bar{\mathbb{B}}}) = \langle \Phi_{\bar{\mathbb{B}}_1}(x_{\bar{\mathbb{B}}_1}) \cdot \ldots \cdot \Phi_{\bar{\mathbb{B}}_n}(x_{\bar{\mathbb{B}}_n}) \rangle_c.$$

Lemma 2. *The number of terms in the sum (6) are at most*

$$2^{\sum_{i=1}^{n}|\ell_i|} \times (Nd+N)^{\sum_{i=1}^{n}|\ell_i|}.$$

Proof. If we fix $\mathbb{B}$ and the length of $|\ell_i|$ of $\mathbb{A}_i$ (i.e. we fix the operator $S_{|\ell|}^{\ell_1,\ldots,\ell_n}$) each term in the sum (6) is in one-to-one correspondence with a sequence $\mathbb{L}$ having length $|\ell_1|+\cdots+|\ell_n|$ extending $\mathbb{B}$.

Indeed, if we choose a sequence $\mathbb{L}\supset\mathbb{B}$ extending $\mathbb{B}$ there is only one way of partitioning $\mathbb{L}$ in the subsequence $\mathbb{A}_i$ since the length $|\ell_i|$ of $\mathbb{A}_i$ is fixed from the beginning. Once chosen $\mathbb{L}$ (and so $\mathbb{A}_i$) the only possibility of partitioning $\mathbb{B}$ with the subsequence $\mathbb{B}_i$, having $\mathbb{B}\subset\mathbb{L}$ fixed, is to take $\mathbb{B}_i=\mathbb{A}_i\cap\mathbb{B}$.

The number of sequences $\mathbb{L}$ extending $\mathbb{B}$ and having length $|\ell_1|+\ldots+|\ell_n|$ is

$$\binom{\sum_{i=1}^{n}|\ell_i|}{|\ell|}(Nd+N)^{\sum_{i=1}^{n}|\ell_i|-|\ell|}\leqslant 2^{\sum_{i=1}^{n}|\ell_i|}\times(Nd+N)^{\sum_{i=1}^{n}|\ell_i|}.$$ ☐

Lemma 3. *We have that*

$$|\mathcal{C}(x_{\bar{\mathbb{B}}})| \leqslant (C_{\mathrm{GH}})^s \sum_{\mathcal{T} \text{ anchored tree for } (\bar{\mathbb{B}}_1,\ldots,\bar{\mathbb{B}}_n)} \prod_{(x,x')\in\mathcal{T}} M(x-x')$$

where

$$s = \frac{1}{2}\left(\sum_{i=1}^{n} |\bar{\mathbb{B}}_i|\right) - (n-1) \leqslant \frac{1}{2}\left(\sum_{i=1}^{n} |\ell_i|\right),$$

$$M(x) = C_{\chi_1} e^{-C_{\chi_2}\frac{|x|^\sigma}{\gamma^\sigma}}$$

and $\mathcal{T}$ is an anchored three for $\bar{\mathbb{B}}$ with respect to the partition $(\bar{\mathbb{B}}_1,\ldots,\bar{\mathbb{B}}_n)$ if $\mathcal{T}$ is a tree with vertexes $\bar{\mathbb{B}}$ such that is a connected tree of the quotient $\bar{\mathbb{B}}/(\bar{\mathbb{B}}_1,\ldots,\bar{\mathbb{B}}_n)$.

Proof. We have that

$$\mathcal{C}_{\bar{\mathbb{B}}}(x_{\bar{\mathbb{B}}}) = \langle \Phi(x_{\bar{\mathbb{B}}_1}|\bar{\mathbb{B}}_1)\cdots\Phi(x_{\bar{\mathbb{B}}_n}|\bar{\mathbb{B}}_n)\rangle_c,$$

where $\langle \cdot_1 \times \cdot_2 \times \cdots \times \cdot_n\rangle_c$ is the reduced expectation. If we apply the Gawedzki-Kupiainen-Lesniewski (GKL) theorem we obtain the statement of the lemma with

$$C_{\mathrm{GH}} = \max\left(\int |\hat{g}(k)|\frac{\mathrm{d}k}{(2\pi)^d}, \int |\hat{g}(k)|^2\frac{\mathrm{d}k}{(2\pi)^d}\right).$$

□

Lemma 4. *We have that*

$$w(x_{\mathbb{B}}) \leqslant \prod_{i=1}^{n} w(x_{\mathbb{A}_i}) \times \prod_{(x,x')\in\mathcal{T}} w(\{x,x'\}).$$

Proof. Let $\mathcal{T}$ be an anchored tree for $\bar{\mathbb{B}}$ with partition $(\bar{\mathbb{B}}_1,\ldots,\bar{\mathbb{B}}_n)$. If $\tau_1,\ldots,\tau_n$ are trees connecting $x_{\mathbb{A}_1},\ldots,x_{\mathbb{A}_n}$ respectively, the tree $\mathcal{T}\cup\tau_1\cup\cdots\cup\tau_n$ connects the point $\{x_{\mathbb{B}},x_{\bar{\mathbb{B}}}\}$ and so $x_{\mathbb{B}}$. This implies that

$$\mathrm{St}(x_{\mathbb{B}}) \leqslant \mathrm{St}(\{x_{\mathbb{B}},x_{\bar{\mathbb{B}}}\}) \leqslant \sum_{i=1}^{n} \mathrm{St}(x_{\mathbb{A}_i}) + \sum_{(x,x')\in\mathcal{T}} |x-x'|.$$

Since $0<\sigma<1$, for any $p_1,\ldots,p_h\in\mathbb{R}_+$ we have

$$\left(\sum_{i=1}^{h} p_i\right)^{\sigma} \leqslant \sum_{i=1}^{h} p_i^{\sigma}.$$

Using the two previous inequalities and the definition of $w(x_{\mathbb{C}})$ as

$$w(x_{\mathbb{C}}) = \exp\left(\bar{C}\left(\frac{\mathrm{St}(x_{\mathbb{C}})}{\gamma}\right)^{\sigma}\right)$$

we get the thesis. □

Lemma 5. *For any* $((\mathbb{B}_i,\mathbb{A}_i))_{i=1,\ldots,n}$ *we have that*

$$\|K_{((\mathbb{B}_i,\mathbb{A}_i))_{i=1,\ldots,n}}\|_w \leqslant \mathcal{N}_{(\bar{\mathbb{B}}_1,\ldots,\bar{\mathbb{B}}_n)}(C_{\mathrm{GH}})^s\|M\|_w^{n-1}\prod_{i=1}^{n}\|h_i\|_w$$

where $s=\sum_{i=1}^n|\ell_i|$ *and* $\mathcal{N}_{(\mathbb{B}_1,\ldots,\mathbb{B}_n)}$ *is the number of anchored trees of* $\bar{\mathbb{B}}$ *with respect to the partition* $(\bar{\mathbb{B}}_1,\ldots,\bar{\mathbb{B}}_n)$.

Proof. First we note that, by Lemma 3 and Lemma 4, we get

$$\begin{aligned}\|K_{((\mathbb{B}_i,\mathbb{A}_i))}\|_w &= \int|K_{((\mathbb{B}_i,\mathbb{A}_i))}(x_{\mathbb{B}})|w(x_{\mathbb{B}})\mathrm{d}x_{\mathbb{B}}\\ &\leqslant \sum_{\mathcal{T}}\int\prod_{(x,x')\in\mathcal{T}}M(x-x')\prod_{i=1}^{n}|h_i(x_{\mathbb{A}_i},\mathbb{A}_i)|w(x_{\mathbb{B}})\mathrm{d}x_{\mathbb{B}\cup\bar{\mathbb{B}}}\\ &\leqslant \sum_{\mathcal{T}}\int\prod_{(x,x')\in\mathcal{T}}w(\{x,x'\})M(x-x')\\ &\quad\times\prod_{i=1}^{n}w(x_{\mathbb{A}})|h_i(x_{\mathbb{A}_i},\mathbb{A}_i)|\mathrm{d}x_{\mathbb{B}\cup\bar{\mathbb{B}}}.\end{aligned}\tag{7}$$

In order to get the thesis we use an “amputating tree leaves” argument.

More precisely, given an anchored tree $\mathcal{T}$ we can find a leaf: i.e. we can consider a $k \in \{1, \ldots, n\}$ and a set $x_{\tilde{\mathbb{B}}_k} \subset x_{\mathbb{A}_k}$ such that $x_{\tilde{\mathbb{B}}_k}$ is connected to the remaining vertexes of $\{x_{\mathbb{B}}, x_{\bar{\mathbb{B}}}\}$ by only one edge, say (z, z') where $z \in x_{\mathbb{A}_k}$ and $x_{\tilde{\mathbb{B}}_k} = x_{\mathbb{A}_k} \setminus z$.

We then integrate the integral (7) first with respect to $x_{\mathbb{A}_k} \setminus z$ (considering z fixed) obtaining a factor of the form

$$\|h_k\|_w = \int |h_k(x_{\mathbb{A}_k}, \mathbb{A}_k)_{z=0}| w(x_{\mathbb{A}_k})_{z=0} \mathrm{d}x_{\tilde{\mathbb{B}}_k}$$

and then we integrate with respect to z obtaining a factor of the form

$$\|M\|_w = \int M(\tilde{z}) w(\tilde{z}) \mathrm{d}\tilde{z}.$$

Repeating the argument $n-1$ times we get the thesis. □

Lemma 6. *We have that*

$$\mathcal{N}_{\bar{\mathbb{B}}_1,\ldots,\bar{\mathbb{B}}_n} \leqslant n! 4^{\sum_{i=1}^n |\ell_i|}.$$

Proof. The proof is a consequence of the (standard) estimate on anchored trees with respect to the partition $(\bar{\mathbb{B}}_1,\ldots,\bar{\mathbb{B}}_n)$. Indeed we have

$$\mathcal{N}_{\bar{\mathbb{B}}_1,\ldots,\bar{\mathbb{B}}_n} \leqslant n! 4^{\sum_{i=1}^n |\bar{\mathbb{B}}_i|} = n! 4^{(\sum_{i=1}^n |\ell_i|) - |\ell|} \leqslant n! 4^{\sum_{i=1}^n |\ell_i|}.$$ □

Proof of inequality (4). By equation (6) and Lemma 2 we have

$$\|S_{|\ell|}^{\ell_1,\ldots,\ell_n}\|_w \leqslant \frac{2^{\sum_{i=1}^n |\ell_i|} \times (Nd+N)^{\sum_{i=1}^n |\ell_i|}}{n!} \sup_{((\mathbb{B}_i,\mathbb{A}_i))} \|K_{((\mathbb{B}_i,\mathbb{A}_i))}\|_w.$$

By Lemma 5 and Lemma 6 we have

$$\|K_{((\mathbb{B}_i,\mathbb{A}_i))}\| \leqslant n! (4\sqrt{C_{\mathrm{GH}}})^{\sum_{i=1}^n |\ell_i|} \|M\|_w^{n-1} \prod_{i=1}^n \|h_i\|_w.$$

Inequality (4) follows by taking $C_\gamma = \|M\|_w$ and $C_0 = 8(Nd+N)\sqrt{C_{\mathrm{GH}}}$. □

In order to prove inequality (5) we need to recall the precise definition of the operators $T_\ell^{|\ell|}$. First we consider $\mathcal{H}_{|\ell|,p}$ a kernel in $B_{|\ell|}$ (i.e. a kernel corresponding to the monomial where the fields Ψ_a (for some $a \in \{1,\ldots,N\}$) appear exactly $|\ell|-p$ times and the derived fields $\partial_\mu\Psi_a$ (for some $a \in \{1,\ldots,N\}$ and $\mu^i \in \{1,\ldots,d\}$) appear exactly p times).

This means that the field $\Psi(\mathcal{H}_{|\ell|,p})$ corresponding to $\mathcal{H}_{|\ell|,p}$ is given by an expression of the form

$$\Psi(\mathcal{H}_{|\ell|,p}) = \sum_{\tilde{\mathrm{A}}=((\mu^i,a^i)),\sum|\mu^i|=p} \int \mathcal{H}_{|\ell|,p}(x_1,\ldots,x_{|\ell|},\tilde{\mathrm{A}})\partial_{\mu^1}\Psi_{a_1}(x_1)\cdots\partial_{\mu^{|\ell|}}\Psi_{a_{|\ell|}}(x_{|\ell|})\mathrm{d}x.$$

The kernels $\mathcal{H}_{|\ell|,p}$ are not uniquely determined since by the commutation properties of the fields Ψ_a and the symmetries of the model. We do not enter into the details of this choice but we suppose to fix a representative of the equivalence class before.

We want to decompose $\mathcal{H}_{|\ell|,p}$ into the local and remaining part when $|\ell|=2,4$.

We start with $\mathcal{H}_{2,1}$ for which we have

$$\Psi(\mathcal{H}_{2,1})=\sum_{a,b}\sum_{\mu\in\{1,\ldots,d\}}\int\mathcal{H}_{2,1}(x,((a,\mu),b))\partial_\mu\Psi_a(x_1)\Psi_b(x_2)\mathrm{d}x_1\mathrm{d}x_2$$

Using Lagrange theorem we can rewrite

$$\Psi_b(x_2)=\Psi_b(x_1)+\int_0^1(\partial_t\Psi_b(x_1+t(x_1-x_2)))\mathrm{d}t.$$

From which we get

$$\begin{aligned}\Psi(\mathcal{H}_{2,1}) \;=\; & \sum_{a,b}\sum_{\mu\in\{1,\ldots,d\}}\int\mathcal{H}_{2,1}(x,((a,\mu),b))\partial_\mu\Psi_a(x_1)\Psi_b(x_1)\mathrm{d}x_1\mathrm{d}x_2\\ & +\sum_{a,b}\sum_{\mu,\nu\in\{1,\ldots,d\}}\int H_{2,2}(x,((a,\mu),(b,\nu)))\partial_\mu\Psi_a(x_1)\partial_\nu\Psi_b(x_2)\mathrm{d}x_1\mathrm{d}x_2,\end{aligned}$$

where $H_{2,2}$ is related to $\mathcal{H}_{2,0}$ by the formula, for any $f\in C_0^\infty$,

$$\int H_{2,2}(x,((a,\mu),(b,\nu)))_{x_1=0}f(x_2)\mathrm{d}x_2=\int\mathcal{H}_{2,1}(x,((a,\mu),b))_{x_1=0}x_2^\nu\int_0^1 f(tx_2)\mathrm{d}t\mathrm{d}x_2$$

Since $\mathcal{H}_{2,1}(x,((a,\mu),b))$ is odd with respect the transformation $x\mapsto -x$ we have that the first integral is 0.

From the second part we get the expression of $T_{2,2}^{2,1}$ namely

$$T_{2,2}^{2,1}(\mathcal{H}_{2,1}) = (H_{2,2}(\cdot,((a,\mu),(b,\nu))))_{(\mu,a),(\nu,b)}$$

where $L_{a,b}\in\mathbb{R}$ are suitable constants.

Consider now the case of $\mathcal{H}_{2,0}$ for which

$$\Psi(\mathcal{H}_{2,0}) = \sum_{a,b}\int \mathcal{H}_{2,0}(x,(a,b))\Psi_a(x_1)\Psi_b(x_2)\mathrm{d}x_1\mathrm{d}x_2.$$

We can use the Lagrange theorem as before obtaining

$$\begin{aligned}\Psi(\mathcal{H}_{2,0}) &= \sum_{a,b}\left(\int \Psi_a(x_1)\Psi_b(x_1)\mathrm{d}x_1\right)\int \mathcal{H}_{2,0}((0,x_2),(a,b))\mathrm{d}x_2 + \\ &\quad - \sum_{a,b,\mu\in\{1,\ldots,d\}}\int \tilde{\mathcal{H}}_{2,1}(x,(a,(b,\mu)))\partial_\mu\Psi_a(x_1)\Psi_b(x_2)\mathrm{d}x_1\mathrm{d}x_2.\end{aligned}$$

The first term in the previous sum defined the local part of the trimming operator namely

$$T^{2,0}_{2L}(\mathcal{H}_{2,0}) = \sum_{a,b} C_{a,b} \int \mathcal{H}_{2,0}((0,x_2),(a,b))\mathrm{d}x_2$$

for suitable constants $C_{a,b} \in \mathbb{R}$. We can apply again the Lagrange formula to what remains obtaining the expression for $T^{2,0}_{2R}(\mathcal{H}_{2,0})$.

For the other terms we use the equality

$$\begin{aligned}\Psi_a(x_1)\Psi_b(x_2)\Psi_c(x_3)\Psi_h(x_4) =& \\ &= \Psi_a(x_1)\Psi_b(x_1)\Psi_c(x_1)\Psi_h(x_1) + \Psi_a(x_1)\int_0^1 \partial_t[\Psi_b(x_2^t)\Psi_c(x_3^t)\Psi_h(x_4^t)]\mathrm{d}t,\end{aligned}$$

where $x_i^t = x_1 + t(x_i - x_1)$.

From this we can write

$$\begin{aligned}\Psi(\mathcal{H}_{4,0}) &= \sum_{a,b,c,h} \int \mathcal{H}_{4,0}(x,(a,b,c,h))\Psi_a(x_1)\cdots\Psi_h(x_4)\mathrm{d}x_1\cdots\mathrm{d}x_4 \\ &= \sum_{a,b,c,h}\left(\int \Psi_a(x_1)\cdots\Psi_h(x_1)\mathrm{d}x_1\right)\int \mathcal{H}_{4,0}(x,(a,b,c,h))_{x_1=0}\mathrm{d}x_2\cdots\mathrm{d}x_4 + \\ &\quad + \sum_{a,\cdots,h,\mu\in\{1,\ldots,d\}} \int H_{4,1}(x,(a,b,\ldots,(h,\mu)))\Psi_a(x_1)\cdots\partial_\mu\Psi_h(x_4)\mathrm{d}x_1\cdots\mathrm{d}x_d\end{aligned}$$

where analogously to the previous case, for any $f \in C_0^\infty$, we have

$$\int H_{4,1}(x,(a,b,\ldots,(h,\mu)))_{x_1=0} f(x_2,x_3,x_4)\mathrm{d}x_2\mathrm{d}x_3\mathrm{d}x_4 =$$
$$= \int \mathcal{H}_{4,0}(x,(a,b,c,h))_{x_1=0}(x_2^\mu - x_3^\mu + x_4^\mu)\int_0^1 f(tx_2,tx_3,tx_4)\mathrm{d}t\mathrm{d}x_2\mathrm{d}x_3\mathrm{d}x_4.$$

In this way we can define the operator $T_{4L}^{4,0}$ and $T_{4R}^{4,0}$ as

$$T_{4L}^{4,0}(\mathcal{H}_{4,0}) = \sum_{a,b,c,h} L'_{a,b,c,h}\int \mathcal{H}_{4,0}(x,(a,b,c,h))_{x_1=0}\mathrm{d}x_2\mathrm{d}x_3\mathrm{d}x_4$$

and

$$T_{4R}^{4,0}(\mathcal{H}_{4,0}) = H_{4,1}.$$

We have the following theorem

Theorem 7. *Consider an Hamiltonian $\mathcal{H}$ and $\gamma > 2$, then there is a constant C_R such that*

$$\|T_{2R}^{2,1}(\mathcal{H}_{2,1})\|_{w(\cdot/\gamma)} \leqslant C_R \gamma \|\mathcal{H}_{2,1}\|_w$$

$$\|T_{2L}^{2,0}(\mathcal{H}_{2,0})\|_{w(\cdot/\gamma)} \leqslant C_R \|\mathcal{H}_{2,0}\|_w$$

$$\|T_{2R}^{2,0}(\mathcal{H}_{2,0})\|_{w(\cdot/\gamma)} \leqslant C_R \gamma^2 \|\mathcal{H}_{2,0}\|_w$$

$$\|T_{4L}^{4,0}(\mathcal{H}_{4,0})\|_{w(\cdot/\gamma)} \leqslant C_R \|\mathcal{H}_{4,0}\|_w$$

$$\|T_{4R}^{4,0}(\mathcal{H}_{4,0})\|_{w(\cdot/\gamma)} \leqslant C_R \gamma \|\mathcal{H}_{4,0}\|_w$$

Proof. Inequalities $\|T_{2L}^{2,0}(\mathcal{H}_{2,0})\|_w \leqslant C_R \|\mathcal{H}_{2,0}\|_w$ and $\|T_{4L}^{4,0}(\mathcal{H}_{4,0})\|_w \leqslant C_R \|\mathcal{H}_{4,0}\|_w$ follows directly from the fact that

$$\|T_{2L}^{2,0}(\mathcal{H}_{2,0})\|_w = \left| \sum_{a,b} C_{a,b} \int \mathcal{H}_{2,0}((0,x_2),(a,b)) \mathrm{d}x_2 \right| \leqslant C_R \|\mathcal{H}_{2,0}\|_w$$

$$\|T_{4L}^{4,0}(\mathcal{H}_{4,0})\|_w = \left| \sum_{a,b,c,h} L'_{a,b,c,h} \int \mathcal{H}_{4,0}(x,(a,b,c,h))_{x_1=0} \mathrm{d}x_2 \mathrm{d}x_3 \mathrm{d}x_4 \right| \leqslant C_R \|\mathcal{H}_{4,0}\|_w.$$

Here we use the fact that $w \geqslant 1$.

Let us consider one of the other cases, for example $\|T_{2R}^{2,1}(\mathcal{H}_{2,1})\|_{w(\cdot/\gamma)} \leqslant C_R \gamma \|\mathcal{H}_{2,1}\|_w$. In this case $T_{2R}^{2,1}(\mathcal{H}_{2,1}) = H_{2,1}$ where $H_{2,1}$ and $\mathcal{H}_{2,1}$ are related by the identity, for any $f \in C^0$,

$$\int H_{2,2}(x, ((a, \mu), (b, \nu)))_{x_1=0} f(x_2) \mathrm{d}x_2 = \int \mathcal{H}_{2,1}(x, ((a, \mu), b))_{x_1=0} x_2^{\nu} \int_0^1 f(t x_2) \mathrm{d}t \mathrm{d}x_2.$$

We can choose $f(x) = f_{a,b,\nu,\mu}(x_2) = \operatorname{sign}(H_{2,2}(0, x_2, (a, \mu), (b, \nu))) w_\gamma(x_2)$ where $w_\gamma(x_2) = w(x_2/\gamma)$ obtaining

$$\begin{aligned}
\|H_{2,2}\|_{w_\gamma} &= \sum_{a,b,\nu,\mu} \int H_{2,2}(x, ((a, \mu), (b, \nu)))_{x_1=0} f_{a,b,\nu,\mu}(x_2) \mathrm{d}x_2 \\
&= \sum_{a,b,\nu,\mu} \int \mathcal{H}_{2,1}(x, ((a, \mu), b))_{x_1=0} x_2^{\nu} \int_0^1 f_{a,b,\nu,\mu}(t x_2) \mathrm{d}t \mathrm{d}x_2 \\
&\leqslant \sum_{a,b,\nu,\mu} \int |\mathcal{H}_{2,1}(x, ((a, \mu), b))_{x_1=0}| |x_2^{\nu}| w(t x_2/\gamma) \mathrm{d}t \mathrm{d}x_2 \\
&\leqslant \sum_{a,b,\nu,\mu} \int |\mathcal{H}_{2,1}(x, ((a, \mu), b))_{x_1=0}| |x_2^{\nu}| w(x_2/\gamma) \mathrm{d}x_2 \\
&\leqslant C_R' \gamma \sum_{a,b,\nu,\mu} \int |\mathcal{H}_{2,1}(x, ((a, \mu), b))_{x_1=0}| w(x_2) \mathrm{d}x_2 \leqslant C_R \gamma \|\mathcal{H}_{2,1}\|_w
\end{aligned}$$

where we used the fact that $w(tk)\leqslant w(k)$ if $0\leqslant t\leqslant 1$, and that there exists a constant $C'_R>0$ not depending on γ such that

$$\sup_{\nu\in\{1,\ldots,d\},x\in\mathbb{R}^d}\frac{|x^\nu|w(x/\gamma)}{\gamma w(x)}=\sup_{\nu,x}\frac{|x^\nu|\exp\Big(\bar{C}\Big(\frac{1}{\gamma^\sigma}-1\Big)\frac{|x|^\sigma}{\gamma^\sigma}\Big)}{\gamma}<C'_R.$$

A similar method can be used to prove the inequalities for $\|T^{4,0}_{4R}(\mathcal{H}_{4,0})\|_{w(\cdot/\gamma)}$ and $\|T^{2,0}_{2R}(\mathcal{H}_{2,0})\|_{w(\cdot/\gamma)}$. In particular in the latter case we obtain a γ^2 since

$$\|T^{2,0}_{2R}(\mathcal{H}_{2,0})\|_{w(\cdot/\gamma)}\leqslant C''\sum_{a,b,\nu}\int|\mathcal{H}_{2,0}(x,(a,b))_{x_1=0}||x_2^\nu|^2w(x_2/\gamma)\mathrm{d}x_2$$

and

$$\sup_{\nu\in\{1,\ldots,d\},x\in\mathbb{R}^d}\frac{|x^\nu|^2w(x/\gamma)}{\gamma^2w(x)}<+\infty.\qquad\square$$

Conclusion of the argument

Summarizing the previous results we have that

$$\|S_{|\ell|}^{\ell_1,\ldots,\ell_n}(h_1,\ldots,h_n)\|_w \leqslant \rho_{|\ell|}(h_1,\ldots,h_n), \quad \left\|T_\ell^{|\ell|}(H_{|\ell|})\right\|_{w(\cdot/\gamma)} \leqslant C_R'' \gamma^{g(|\ell|)} \|H_{|\ell|}\|_w \tag{8}$$

where $C_R'' > 0$ and $g\colon \mathbb{R}_+ \to \mathbb{N}_0$ are suitable functions. These two inequalities implies the result for $R_\ell^{\ell_1,\ldots,\ell_n}$. Indeed when $\ell \notin \{2L, 2R, 4L, 4R\}$ then $R_\ell^{\ell_1,\ldots,\ell_n} = D \circ \begin{cases} S_{|\ell|}^{\ell_1,\ldots,\ell_n} \\ 0 \end{cases}$, and thus the thesis follows from inequality (8) and the definition of the operator D. In all the other cases $R_\ell^{\ell_1,\ldots,\ell_n} = D \circ T_\ell^{|\ell|} \circ S_{|\ell|}^{\ell_1,\ldots,\ell_n}$ and thus

$$\begin{aligned}
\|R_\ell^{\ell_1,\ldots,\ell_n}(h_1,\ldots,h_n)\|_w &= \left\|D \circ T_\ell^{|\ell|} \circ S_{|\ell|}^{\ell_1,\ldots,\ell_n}(h_1,\ldots,h_n)\right\|_w \\
&= \gamma^{\alpha_\ell} \left\|T_\ell^{|\ell|} \circ S_{|\ell|}^{\ell_1,\ldots,\ell_n}(h_1,\ldots,h_n)\right\|_{w(\cdot/\gamma)} \\
&\leqslant C_R'' \gamma^{\alpha_\ell + g(|\ell|)} \|S_{|\ell|}^{\ell_1,\ldots,\ell_n}(h_1,\ldots,h_n)\|_w \\
&\leqslant \gamma^{-D_{|\ell|}} \rho_{|\ell|}(h_1,\cdots,h_n).
\end{aligned}$$

It is important to note that $\alpha_{2R} = D_2 - 2$ and $g(2) = 2$ and so $\alpha_{2R} + g(2) = D_2$. We have also $\alpha_{4R} = D_4 - 1$ and $g(4) = 1$ and thus $\alpha_{4R} + g(4) = D_4$.

The argument presented before can be extended also to the (Frechet) derivatives of the (multi-linear) operator $R_\ell^{\ell_1,...,\ell_n}$. Indeed, since the map $R_\ell^{\ell_1,...,\ell_n}$ is multilinear and bounded (for the result that we proved before), it also Frechet differentiable with the Frechet derivatives given by the operator

$$\delta h \mapsto \nabla R_\ell^{\ell_1,...,\ell_n}(h_1,\dots,h_n)[\delta h] = \sum_{i=1}^{n} R_\ell^{\ell_1,...,\ell_n}(h_1,\dots,h_{i-1},\delta h,h_{i+1},\dots,h_n).$$

In this way we get that

$$\|\nabla R_\ell^{\ell_1,...,\ell_n}(h_1,\dots,h_n)\|_{\mathcal{L}(B_\ell,B_\ell)} \leqslant n\gamma^{-D_{|\ell|}}(\rho_{|\ell|}(h_1,\dots,h_n)\wedge 1).$$

In conclusion we proved the following theorem.

Theorem 8. *Suppose that* $\gamma > 2$ *then we have the following inequalities*

$$\|R_\ell^\ell(H_\ell)\|_w \leqslant \begin{cases} \gamma^{-D_2-2}\|H_{2R}\|_w \; for \; \ell = 2R \\ \gamma^{-D_4-1}\|H_{4R}\|_w \; for \; \ell = 4R \\ \gamma^{-D_\ell}\|H_\ell\|_w \qquad for \, \ell \geqslant 6 \end{cases}$$

and

$$\|R_\ell^{\ell_1,\dots,\ell_n}(h_1,\cdots,h_n)\|_w \leqslant \gamma^{-D_\ell}\rho_{|\ell|}(h_1,\cdots,h_n)$$

with

$$\rho_{|\ell|}(h_1,\cdots,h_n) = \begin{cases} C_\gamma^{n-1}\prod_{i=1}^n C_0^{|\ell_i|}\|h_i\|_w \; if \; \sum_i |\ell_i| \geqslant |\ell| + 2(n-1) \\ 0 \qquad\qquad\qquad\qquad otherwise \end{cases}.$$

Finally, similar estimates hold for $\nabla R_\ell^{\ell_1,\dots,\ell_n}$.