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Bounds for Fermionic expectations

M. Gubinelli – University of Bonn & HCM

Goal: Obtain effective representations for the RG map

Outline

- Reminder on the RG map
- Connected expectations and the iteration on the kernels
- The BBF formula
- The Gawedzki-Kupiainen-Lesniewski (GKL) bound and the Gram bound

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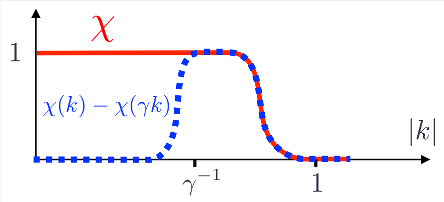
Decomposition of the propagator

$$P = P_1 = P_\gamma + g, \quad \hat{P}_\gamma(k) = \frac{\chi(\gamma k)}{|k|^{d/2+\varepsilon}},$$



$$\hat{g}(k) = \frac{\chi(k) - \chi(\gamma k)}{|k|^{d/2+\varepsilon}}$$

Rescaling

$$P_\gamma(x) = \int \frac{\chi(\gamma k)}{|k|^{d/2+\varepsilon}} e^{ik\cdot x} \mathrm{d}k = \gamma^{-2[\psi]} P(\gamma^{-1}x), \quad [\psi] = \frac{d}{4} - \frac{\varepsilon}{2}$$

$$(\psi)_\gamma(x) = \gamma^{-[\psi]}\psi(\gamma^{-1}x), \qquad \langle(\psi)_\gamma(x)(\psi)_\gamma(y)\rangle_P = \gamma^{-2[\psi]} P(\gamma^{-1}(x-y)) = P_\gamma(x-y)$$

RG map: $H \to H'$

$$e^{H'(\psi)} = e^{H_{\text{eff}}((\psi)_\gamma)} = \int e^{H((\psi)_\gamma + \phi)} \mu_g(\mathrm{d}\phi), \qquad \psi \sim P.$$

$$\int O((\psi)_\gamma) e^{H'(\psi)} \mu_P(\mathrm{d}\psi) = \int \mu_P(\mathrm{d}\psi) O((\psi)_\gamma) e^{H_{\text{eff}}((\psi)_\gamma)}$$

$$= \int \mu_P(\mathrm{d}\psi) \int O((\psi)_\gamma) e^{H((\psi)_\gamma + \phi)} \mu_g(\mathrm{d}\phi)$$

$$= \int \mu_{P_\gamma}(\mathrm{d}\psi) \int O(\psi) e^{H(\psi + \phi)} \mu_g(\mathrm{d}\phi) = \int O(\psi) e^{H(\psi)} \mu_P(\mathrm{d}\psi)$$

provided $O((\psi)_\gamma) = O((\psi)_\gamma + \varphi)$

Compute the log

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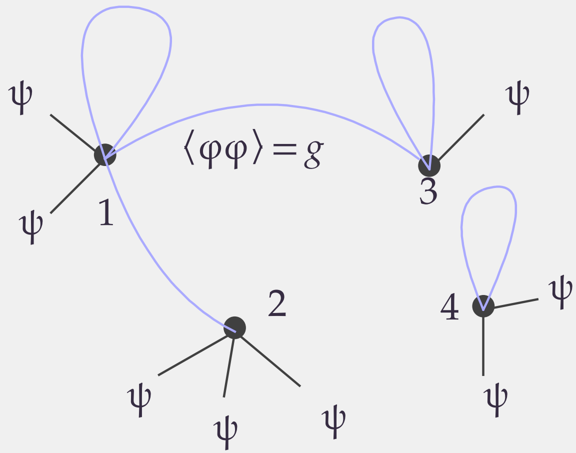
$$H_{\text{eff}}(\psi) = \log \int e^{H(\psi+\phi)} \mu_g(d\phi)$$

$$= \log \sum_{n \geqslant 0} \frac{1}{n!} \Big\langle \underbrace{H(\psi+\phi) \cdots H(\psi+\phi)}_{n} \Big\rangle_{\phi}$$

$$= \log \sum_{n \geqslant 0} \frac{1}{n!} \sum_{\Gamma} \{\text{Wick contractions for } \Gamma\}$$



$$\Pi = \{\{1,2,3\},\{4\}\}$$

$\triangleright\Gamma$ is a possible set of Wick contraction, it give rise to a graph over n vertices

$$H_{\text{eff}}(\psi) = \log \sum_{n\geqslant 0} \frac{1}{n!} \sum_{\Pi} \sum_{\Gamma} \{\text{Wick contractions for } \Gamma \text{ compatible with } \Pi\}$$

$\triangleright$Each graph has k connected components $\Gamma=\Gamma_1\cdots\Gamma_k$. We split the sum according to the associated partition Π of the n vertices. We denote $\Gamma\ll\Pi$ the compatibility relation.

$$H_{\mathrm{eff}}(\psi)=\log\sum_{n\geqslant 0}\frac{1}{n!}\sum_{\Pi}\sum_{\Gamma:\Gamma\ll\Pi}\prod_{j=1,\ldots,k}\{\text{Connected Wick contractions for }\Gamma_j\}$$

$\triangleright$Given k connected graphs $\Gamma_1,\ldots,\Gamma_k$ on $n_1,\ldots,n_k$ vertices $H(\psi+\varphi)$ we have

$$\frac{n!}{k!n_1!\cdots n_k!}$$

graphs Γ on the $n=n_1+\cdots+n_k$ vertices with the same set of connected components.

Therefore

$$H_{\rm eff}(\psi) = \log \sum_k \frac{1}{k!} \sum_{\Gamma_1,\ldots,\Gamma_k} \frac{1}{n_1!\cdots n_k!} \prod_j \{\text{conn. Wick contractions for } \Gamma_j\}$$

$$= \log \sum_k \frac{1}{k!} \left[\sum_{\Gamma_1} \frac{1}{n_1!} \{\text{conn. Wick contractions for } \Gamma_1\} \right]^k$$

$$= \sum_{\Gamma_1} \frac{1}{n_1!} \{\text{conn. Wick contractions for } \Gamma_1\}$$

Finally we can take the log (all this sums are finite for finitely many Grassmann vars)

$$\boxed{H_{\rm eff}(\psi) = \sum_{n \geqslant 1} \frac{1}{n!} \left\langle H(\psi+\phi); \cdots ; H(\psi+\phi) \right\rangle_{\phi,c}}$$

In particular we had

$$\Big\langle \underbrace{H(\psi+\phi)\cdots H(\psi+\phi)}_{n} \Big\rangle_{\phi} = \sum_{\Gamma} \{\text{Wick contractions for } \Gamma\}$$

$$= \sum_{\Pi} \sum_{\Gamma:\Gamma \ll \Pi} \prod_{j=1,\dots,k} \{\text{Wick contractions for } \Gamma_j\} = \sum_{\Pi} \prod_{j=1,\dots,k} \Big\langle \underbrace{H(\psi+\phi);\cdots;H(\psi+\phi)}_{n_j} \Big\rangle_{\phi,c}$$

One can use this formula recursively to express the connected functions:

$$\langle H(\psi+\varphi)\rangle_{\phi} = \langle H(\psi+\varphi)\rangle_{\phi,c}$$

$$\langle H(\psi+\varphi)H(\psi+\varphi)\rangle_{\phi} = \langle H(\psi+\varphi);H(\psi+\varphi)\rangle_{\phi,c} + \langle H(\psi+\varphi)\rangle_{\phi,c}\langle H(\psi+\varphi)\rangle_{\phi,c}$$

$$\langle H(\psi+\varphi)H(\psi+\varphi)H(\psi+\varphi)\rangle_{\phi} = \langle H(\psi+\varphi);H(\psi+\varphi);H(\psi+\varphi)\rangle_{\phi,c}$$

$$+3\langle H(\psi+\varphi);H(\psi+\varphi)\rangle_{\phi,c}\langle H(\psi+\varphi)\rangle_{\phi,c} + \langle H(\psi+\varphi)\rangle_{\phi,c}\langle H(\psi+\varphi)\rangle_{\phi,c}\langle H(\psi+\varphi)\rangle_{\phi,c}$$

Et cetera...

RG map (II)

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Notations: Monomials

$$A_k = a_k, (a_k, \mu_k), \boldsymbol{A} = (A_1, \dots, A_m), \quad \Phi_{\boldsymbol{A}}(\boldsymbol{x}) = \phi_{A_1}(x_1) \cdots \phi_{A_m}(x_m), \quad \phi_{(a,\mu)}(x) = \partial_\mu \phi_a(x)$$

Interactions

$$H(\psi) = \sum_{\boldsymbol{A}} \int \mathrm{d}\boldsymbol{x}_{\boldsymbol{A}}\, H_{\boldsymbol{A}}(\boldsymbol{x}_{\boldsymbol{A}}) \Psi_{\boldsymbol{A}}(\boldsymbol{x}_{\boldsymbol{A}})$$

Simple expectations

$$\langle \Phi_{A_1}(x_1) \cdots \Phi_{A_{2n}}(x_{2n}) \rangle = \sum (-1)^\pi \underbrace{\langle \Phi_{A_{\pi(1)}}(x_{\pi(1)}) \Phi_{A_{\pi(2)}}(x_{\pi(2)}) \rangle \cdots \langle \Phi_{A_{\pi(2n-1)}}(x_{\pi(2n-1)}) \Phi_{A_{\pi(2n)}}(x_{\pi(2n)}) \rangle}_{\text{Wick contractions}}$$

Connected expectations

$$\langle \Phi_{A_1}(x_1) \cdots \Phi_{A_{2n}}(x_{2n}) \rangle_c = \sum (-1)^\pi \underbrace{\langle \Phi_{A_{\pi(1)}}(x_{\pi(1)}) \Phi_{A_{\pi(2)}}(x_{\pi(2)}) \rangle \cdots \langle \Phi_{A_{\pi(2n-1)}}(x_{\pi(2n-1)}) \Phi_{A_{\pi(2n)}}(x_{\pi(2n)}) \rangle}_{\text{Only connected Wick contractions}}$$

Using the expansion with the kernels $\{H_A(\boldsymbol{x})\}$ we have

$$H(\psi+\varphi)=\sum_{A}\sum_{B\subseteq A}\int \mathrm{d}\boldsymbol{x}_A\, H_A(\boldsymbol{x}_A)\Psi_B(\boldsymbol{x}_B)\Phi_{\bar{B}}(\boldsymbol{x}_{\bar{B}}),$$

with $\bar{\boldsymbol{B}}=\boldsymbol{A}\setminus\boldsymbol{B}$.

The new kernels for H_{eff} (before rescaling) are given by

$$H_{\mathrm{eff},B}(\boldsymbol{x}_B)=\sum_{n}\frac{1}{n!}\sum_{\substack{B_1,\dots,B_n\\ B_1+\cdots+B_n=B}}\sum_{\substack{A_1,\dots,A_n\\ A_k\supset B_k}}(-1)^{\#}\int \mathrm{d}\boldsymbol{x}_{\bar{B}}\prod_{k=1}^{n}H_{A_k}(\boldsymbol{x}_{A_k})\left\langle\prod_{k=1}^{n}\Phi_{\bar{B}_k}(\boldsymbol{x}_{\bar{B}_k})\right\rangle_c$$

This is the RG map on the kernels and we want “good” bounds for the quantity $\langle\prod_{k=1}^{n}\Phi_{\bar{B}_k}(\boldsymbol{x}_{\bar{B}_k})\rangle_c$ in the r.h.s.

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We need a more efficient expression $(\phi_{a_1}(x_1), \phi_{a_2}(x_2), \phi_{a_1}(x_1), \phi_{a_2}(x_2)) = (\eta_1, \eta_2, \eta_3, \eta_4)$

$$\left\langle \prod_{k=1}^{n} \Phi_{A_k}(\boldsymbol{x}_k) \right\rangle = \det(\mathscr{M}) = \int \mathrm{d}\eta \mathrm{d}\bar{\eta}\, e^V, \qquad V = \sum_{i,j} \eta_i \bar{\eta}_j \mathscr{M}_{i,j}, \qquad \mathscr{M}_{i,j} = \langle \phi_{A_i}(x_i) \bar{\phi}_{A_j}(x_j) \rangle,$$

$$V = \frac{1}{2} \sum_{k,l=1}^{n} V_{kl}, \qquad V_{kl} = \sum_{i,j: x_i \in \boldsymbol{x}_k, x_j \in \boldsymbol{x}_l} \eta_i \bar{\eta}_j \mathscr{M}_{i,j} + (k \leftrightarrow l)$$

$$X \subseteq \{1, \ldots, n\}, \qquad V(X) = \frac{1}{2} \sum_{k,l \in X} V_{kl}, \qquad \psi(X) = e^{V(X)}.$$

$$\psi_c(\{k\}) = \psi(\{k\}) = e^{\frac{1}{2} V_{k,k}}, \qquad \psi(X) = \sum_{\Pi \in \mathrm{Part}(X)} \prod_{Y \in \Pi} \psi_c(Y)$$

On the one hand

$$\left\langle \prod_{k=1}^{n} \Phi_{A_k}(\boldsymbol{x}_k) \right\rangle = \int \mathrm{d}\eta \mathrm{d}\bar{\eta}\, \psi(\{1,\ldots,n\}) = \sum_{\Pi \in \mathrm{Part}(X)} (-1)^{\#} \prod_{Y \in \Pi} \int_Y \mathrm{d}\eta \mathrm{d}\bar{\eta}\, \psi_c(Y)$$

and on the other

$$\left\langle \prod_{k=1}^{n} \Phi_{A_k}(\boldsymbol{x}_k) \right\rangle = \sum_{\Pi \in \mathrm{Part}(X)} (-1)^{\#} \prod_{Y \in \Pi} \left\langle \prod_{k \in Y} \Phi_{A_k}(\boldsymbol{x}_k) \right\rangle_c$$

so

$$\left\langle \prod_{k \in Y} \Phi_{A_k}(\boldsymbol{x}_k) \right\rangle_c = (-1)^{\#} \int_Y \mathrm{d}\eta \mathrm{d}\bar{\eta}\, \psi_c(Y)$$

[ ]

Better representation for $\psi_c(X)$. We can take $V_{k,k}=0$ (factor away).

Recall that, recursively,

$$\psi(X)=\psi_c(X), \quad |X|=1, \qquad \psi(X)=\sum_{\Pi\in\mathrm{Part}(X)}\prod_{Y\in\Pi}\psi_c(Y).$$

It follows from

$$\psi(X)=\sum_{Y\ni 1}\psi(X\setminus Y)\psi_c(Y), \qquad \Rightarrow \text{decoupling.}$$

Let $F(X)=\sum_{e\in E(X)}F(e)$ and write $e\dashv Z$ if the edge e connects Z and $X\setminus Z$.

Interpolate

$$F\begin{Bmatrix} s \\ Z \end{Bmatrix}(e)=\begin{cases} sF(e) & \text{if } e\dashv Z \\ F(e) & \text{otherwise} \end{cases}, \qquad F\begin{Bmatrix} s \\ Z \end{Bmatrix}(X)=\sum_{e\in E(X)}F\begin{Bmatrix} s \\ Z \end{Bmatrix}(e)$$

Decoupling Z:

$$F\left\{\begin{matrix}1\\Z\end{matrix}\right\}(X)=F(X), \qquad F\left\{\begin{matrix}0\\Z\end{matrix}\right\}(X)=F(X\setminus Z)+F(Z)$$

$$e^{F(X)}=e^{F\left\{\begin{smallmatrix}0\\Z\end{smallmatrix}\right\}(X)}+\int_0^1 \mathrm{d}s\partial_s e^{F\left\{\begin{smallmatrix}s\\Z\end{smallmatrix}\right\}(X)}=e^{F(X\setminus Z)+F(Z)}+\sum_{e\dashv Z}\int_0^1 F(e)e^{F\left\{\begin{smallmatrix}s\\Z\end{smallmatrix}\right\}(X)}\mathrm{d}s$$

Contine decoupling w.r.t $Z\sqcup e$, … Start with $Z_0=\{1\}$, $Z_1=Z_0\sqcup e_1$,...

$$e^{V(X)}=e^{V(X\setminus Z_0)}+\sum_{e_1\dashv Z_0} e^{V(X\setminus Z_1)}\underbrace{\int_0^1 \mathrm{d}s_1 V(e_1)e^{V\left\{\begin{smallmatrix}s_1\\Z_0\end{smallmatrix}\right\}(X)}}_{=:\psi_c(Z_1)}$$

$$+\sum_{e_1\dashv Z_0}\sum_{e_2\dashv Z_1}\int_0^1 \mathrm{d}s_1\int_0^1 \mathrm{d}s_2 V(e_1)V\left\{\begin{matrix}s_1\\Z_0\end{matrix}\right\}(e_2)e^{V\left\{\begin{smallmatrix}s_1\\Z_0\end{smallmatrix}\right\}\left\{\begin{smallmatrix}s_2\\Z_1\end{smallmatrix}\right\}(X)}$$

$$e^{V(X)} = e^{V(X\setminus Z_0)} + \sum_{e_1 \dashv Z_0} e^{V(X\setminus Z_1)} \underbrace{\int_0^1 \mathrm{d}s_1 V(e_1) e^{V\left\{\substack{s_1 \\ Z_0}\right\}(X)}}_{=:\psi_c(Z_1)}$$

$$+ \sum_{e_1 \dashv Z_0} \sum_{e_2 \dashv Z_1} \int_0^1 \mathrm{d}s_1 \int_0^1 \mathrm{d}s_2 V(e_1) V\begin{Bmatrix} s_1 \\ Z_0 \end{Bmatrix}(e_2) e^{V\left\{\substack{s_1 \\ Z_0}\right\}\left\{\substack{s_2 \\ Z_1}\right\}(X)}$$

$$+ \sum_{e_1 \dashv Z_0} \sum_{e_2 \dashv Z_1} \sum_{e_3 \dashv Z_2} e^{V(X\setminus Z_2)} \int_0^1 \mathrm{d}s_1 \int_0^1 \mathrm{d}s_2 \int_0^1 \mathrm{d}s_3 V(e_1) V\begin{Bmatrix} s_1 \\ Z_0 \end{Bmatrix}(e_2) V\begin{Bmatrix} s_1 \\ Z_0 \end{Bmatrix}\begin{Bmatrix} s_2 \\ Z_1 \end{Bmatrix}(e_3) e^{V\left\{\substack{s_1 \\ Z_0}\right\}\left\{\substack{s_2 \\ Z_1}\right\}\left\{\substack{s_3 \\ Z_2}\right\}(X)}$$

In general:

$$\psi_c(Z) = \sum_{e_1 \dashv Z_0} \cdots \sum_{e_3 \dashv Z_2} \mathbb{1}_{Z=Z_k} \int_0^1 \mathrm{d}s_1 \cdots \int_0^1 \mathrm{d}s_k V(e_1) \cdots V\begin{Bmatrix} s_1 \\ Z_0 \end{Bmatrix} \cdots \begin{Bmatrix} s_{k-1} \\ Z_{k-2} \end{Bmatrix}(e_k) e^{V\left\{\substack{s_1 \\ Z_0}\right\}\cdots\left\{\substack{s_k \\ Z_{k-1}}\right\}(X)}$$

$$e^{V(X)} = \sum_{Y \ni 1} e^{V(X\setminus Y)} \psi_c(Y)$$

$$\int_0^1 \mathrm{d}s_1 \cdots \int_0^1 \mathrm{d}s_k V(e_1) \cdots V\begin{Bmatrix} s_1 \\ Z_0 \end{Bmatrix} \cdots \begin{Bmatrix} s_{k-1} \\ Z_{k-2} \end{Bmatrix}(e_k) e^{V\begin{Bmatrix} s_1 \\ Z_0 \end{Bmatrix} \cdots \begin{Bmatrix} s_k \\ Z_{k-1} \end{Bmatrix}(X)}$$

$$= \prod_{l=1}^{k} V(e_l) \int_0^1 \mathrm{d}s_1 \cdots \int_0^1 \mathrm{d}s_k \underbrace{f(\boldsymbol{s})}_{\geqslant 0} e^{\sum_e r(e) V(e)},$$

$r(e) = s_k \cdots s_{l-1}, \qquad k < l$

$$= \prod_{l=1}^{k} V(e_l) \int \mu(\mathrm{d}r) e^{\sum_e r(e) V(e)},$$

Reorganize wrt. $T = e_1 \sqcup \cdots \sqcup e_{k-1}$ spanning tree of $Z = Z_k \ni 1$..

BBF formula:

$$\psi_c(Z) = \sum_T \prod_{e\in T} V(e) \int \mu_T(\mathrm{d}\boldsymbol{r}) e^{\sum_e r(e)V(e) + \frac{1}{2}\sum_{k\in Z} V_{kk}}$$

- For each $r \in \operatorname{supp} \mu_T$ we have $r(e) = s_k \cdots s_l$ for some $k < l$ and numbers $s_j \in [0,1]$.
- The measure μ_T is a probability measure, i.e.

$$\int \mu_T(\mathrm{d}\boldsymbol{r}) = 1.$$

Proof: Take $V(e) = \varepsilon \mathbb{1}_{e\in T'}$ for $\varepsilon \ll 1$, then from BBF:

$$\psi_c(\{1,\ldots,n\}) = \varepsilon^n \int \mu_{T'}(\mathrm{d}\boldsymbol{r}) + o(\varepsilon^n)$$

and otherwise

$$\psi_c(\{1,\ldots,n\}) = \varepsilon^n + o(\varepsilon^n).$$

Remark: There is another standard formula

$$\psi_c(Z) = \sum_{G \in \text{Conn, graph on } Z} \prod_{(k,l) \in G} (e^{V_{k,l}} - 1) \prod_{k \in Z} e^{\frac{1}{2} V_{k,k}}$$

but the BBF is better since the number of trees on n points grows only as

$$n^{n-2}$$

while the number of graphs as

$$2^{\binom{n}{2}}$$

(for large n almost all graphs are connected).

The BBF is also true for Bosonic models.

Gawedzki-Kupiainen-Lesniewski (GKL) bound

[ ]

$$\left\langle \prod_{k\in X} \Phi_{A_k}(\boldsymbol{x_k}) \right\rangle_c = \int_Y \mathrm{d}\eta \mathrm{d}\bar{\eta}\, \psi_c(X)$$

$$= \sum_T \prod_{e\in T} V(e) \int \mu_T(\mathrm{d}\boldsymbol{r}) \left[\int_X \mathrm{d}\eta \mathrm{d}\bar{\eta} e^{\sum_e r(e)V(e) + \frac{1}{2}\sum_{k\in Z} V_{kk}} \right]$$

$$= \sum_T \prod_{(i,j)\in T} \Gamma_{i,j}(x_i - x_j) \int \mu_T(\mathrm{d}\boldsymbol{r}) \det \mathcal{N}(r)$$

where we can show that

$$\mathcal{N}_{i,j}(r) = r(i,j)\mathcal{M}_{i,j} = \langle u_i, u_j \rangle \langle f_i, h_j \rangle = \langle u_i \otimes f_i, u_j \otimes h_j \rangle$$

$$|\det \mathcal{N}(r)| = |\det(\langle F_i, H_j \rangle)_{i,j}|$$

Gram bound

[ ]

$$|\det(\langle f_i, h_j\rangle_H)_{i,j}| \leqslant \prod_{i,j} \|f_i\|_H \|h_j\|_H$$

Fock space $\Gamma_a(H) = \oplus_{n\geqslant 0} \bigwedge^n H$, Creation operator $a^*(f) f_1 \cdots f_n = f f_1 \cdots f_n$.

$$a(f) f_1 \cdots f_n = \sum_k (-1)^{k-1} \langle f, f_k\rangle (f_1 \cdots \not{f_k} \cdots f_n), \qquad \{a(f), a^*(g)\} = \langle f, g\rangle$$

$$\langle a^*(f_1) \cdots a^*(f_n) 1, a^*(h_1) \cdots a^*(h_n) 1\rangle$$

$$= \sum_k (-1)^{k-1} \langle f_1, h_k\rangle \langle a^*(f_2) \cdots a^*(f_n) 1, a^*(h_1) \cdots a^*(\not{h_k}) \cdots a^*(h_n) 1\rangle$$

$$= \cdots = \det(\langle f_i, h_j\rangle)_{i,j}$$

Now

$$\langle a^*(f)\varphi, a^*(f)\varphi\rangle + \langle a(f)\varphi, a(f)\varphi\rangle = \langle \varphi, (a(f)a^*(f) + a^*(f)a(f))\varphi\rangle = \langle f, f\rangle\langle \varphi, \varphi\rangle$$

so

$$\|a^*(f)\varphi\|^2 + \|a(f)\varphi\|^2 \leqslant \|f\|^2\|\varphi\|^2 \Rightarrow \|a^*(f)\|, \|a(f)\| \leqslant \|f\|$$

and therefore

$$|\det(\langle f_i, h_j\rangle)_{i,j}| \leqslant |\langle a^*(f_1)\cdots a^*(f_n)1, a^*(h_1)\cdots a^*(h_n)1\rangle|$$

$$\leqslant \prod_{i,j} \|a^*(f_i)\|\,\|a^*(h_j)\| \leqslant \prod_{i,j} \|f_i\|\,\|h_j\|$$

In our case

$$\Gamma(x_i - x_j) = \int \frac{\mathrm{d}k}{(2\pi)^d} \frac{\chi(\gamma k) - \chi(k)}{|k|^{d/2+\varepsilon}} e^{ik\cdot(x_i - x_j)} = \langle f_i, h_j \rangle_{L^2(\mathbb{R}^d)}$$

with $g(k) = (\chi(\gamma k) - \chi(k)) / |k|^{d/2+\varepsilon}$ and

$$f_i(k) = \frac{g(k)}{|g(k)|^{1/2}} e^{-ik\cdot x_i}, \qquad h_j(k) = |g(k)|^{1/2} e^{-ik\cdot x_j}$$

so for $\mathrm{Re}\,\varepsilon < d/6$.

$$G_{\mathrm{GH}}^{1/2} = \|f_i\| = \|h_j\| = \left(\int \frac{\mathrm{d}k}{(2\pi)^d} |g(k)|\right)^{1/2} \lesssim \left(\int_{C \lesssim |k| \leqslant 1} \frac{\mathrm{d}k}{|k|^{d/2+\mathrm{Re}\,\varepsilon}}\right)^{1/2} < \infty$$

One can take $\|u_i\| = \|u_j\| \leqslant 1$, so, m total # of points and n vertices in the tree T:

$$|\det \mathcal{N}(r)| \leqslant (G_{\mathrm{GH}})^s, \qquad s = \frac{1}{2}(m - 2(n-1)).$$

Lemma. ***GKL bound***

$$\left|\left\langle\prod_{k\in Y}\Phi_{A_k}(\boldsymbol{x}_k)\right\rangle_c\right|\leqslant(G_{\mathrm{GH}})^s\sum_{\mathcal{T}}\prod_{\text{along }\mathcal{T}}|\Gamma(x_i-x_j)|$$

where $s=\frac{1}{2}\sum_{k\in Y}|\boldsymbol{x}_k|-(|Y|-1)$.

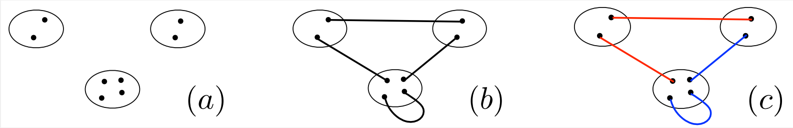

Figure D.1: This illustrates the $n=3$ case of the connected expectation. (a) Three groups of points. (b) A particular connected Wick contraction. (c) Red: an anchored tree consisting of $n-1$ propagators. Blue: remaining s propagators.

[ ]

The number of anchored trees on n groups of points $\boldsymbol{x_1}, \ldots, \boldsymbol{x_n}$ is

$$N_{\mathcal{T}} \leqslant n! 4^{\sum_{k=1}^{n} |\boldsymbol{x_k}|}.$$

Indeed: one has (admit)

$$\frac{(n-1)!}{(d_1-1)! \cdots (d_n-1)!}$$

labelled trees with specified degrees $d_1, \ldots, d_n$ at each vertex. For each edge (k, l) one has at most $|x_k| * |x_l|$ propagators, so in total at most $\prod_k |x_k|^{d_k}$

$$N_{\mathcal{T}} \leqslant (n-1)! \sum_{d_1, \ldots, d_n} \prod_k \frac{|x_k|^{d_k}}{(d_k-1)!} \leqslant n! \prod_k C^{|x_k|}.$$