1 Construction of Gaussian rough path

ASSUMPTION 1.1. $(X_t)_{t\in[0,T]} := \left(X_t^1,\dots,X_t^d\right)_{t\in[0,T]}$ *is centred, continuous Gaussian process with* $X^i \perp X^j$ *for all* $i \neq j$.

We define the **rectangular increments of covariance** as

$$R_X\begin{pmatrix} s,t \\ s',t' \end{pmatrix} := \left(\mathbb{E}\left(X^i_{s,t}X^j_{s',t'}\right)\right)_{i,j=1}^d$$

and for $I,\tilde{I}\subset\mathbb{R}$ and $A: I^2\times\tilde{I}^2\to\mathbb{R}^{d\times d}$, we define

$$\|A\|_{\varrho;I\times\tilde{I}} := \left(\sup_{\substack{\mathcal{P}\subset I\\ \mathcal{P}'\subset\tilde{I}}}\sum_{\substack{[s,t]\in\mathcal{P}\\ [s',t']\in\mathcal{P}'}}\left|A\begin{pmatrix} s,t \\ s',t' \end{pmatrix}\right|^{\varrho}\right)^{\frac{1}{\varrho}}.$$

THEOREM 1.2 (Existence of GRP, [FH20, Theorem 10.4])**.** *If there exists* $\varrho\in\left[1,\frac{3}{2}\right)$ *such that*

$$\|R_{X_i}\|_{\varrho;[s,t]} \lesssim |t-s|^{\frac{1}{\varrho}},\quad \forall 0\le s\le t\le T,\ 1\le i\le d,$$

then with probability one, there exists Gaussian rough path $\mathbf{X} := (X,\mathbb{X})\in C_g^{\alpha}\left([0,T],\mathbb{R}^d\right)$ *for all* $\alpha\in\left(\frac{1}{3},\frac{1}{2\varrho}\right)$.

2 Exponential integrability of GRP

THEOREM 2.1 (Generalised Fernique theorem, [FH20, Theorem 11.7])**.** *Assume* $(E,\mathcal{H},\mu)$ *is an abstract Wiener space. Let* $a,\sigma\in(0,\infty)$ *and consider measurable maps* $f,g: E\to[0,\infty]$ *such that*

1.
$$\mu\left(\{x: g(x)\le a\}\right) > 0.$$

2. *There exists a null-set* N *such that*
$$f(x)\le g(x-h)+\sigma\|h\|_{\mathcal{H}},\quad \forall x\in N^c,\ h\in\mathcal{H}.$$

Then $f(\cdot)$ *has Gaussian tail, more precisely, there exists* $\eta>0$ *such that*

$$\mathbb{E}\left(\exp\left(\eta\,|f(x)|^2\right)\right)\gamma(\mathrm{d}x) < \infty.$$

THEOREM 2.2 ([FH20, Theorem 11.9])**.** *Under the condition of Theorem 1.2, there exists* $\eta>0$ *such that*

$$\mathbb{E}\left(\eta\exp\left(\|\mathbf{X}\|_{\alpha}^2\right)\right) < \infty,$$

where $\mathbf{X} := (X,\mathbb{X})\in C_g^{\alpha}\left([0,T],\mathbb{R}^d\right)$ *as in Theorem 1.2.*

References

[FH20] Peter K Friz and Martin Hairer. *A course on rough paths.* Springer, 2020.