1 Gaussian rough path

1.1 Introduction

ASSUMPTION 1.1. $(X_t)_{t\in[0,T]} := \left(X_t^1, \ldots, X_t^d\right)_{t\in[0,T]}$ *is centred, continuous Gaussian process with* $X^i \perp X^j$ *for all* $i \neq j$.

The law of X is fully determined by its covariance function

$$\begin{aligned} R : [0,T]^2 &\to \mathbf{R}^{d\times d} \\ (s,t) &\mapsto \mathbf{E}\left[X_s \otimes X_t\right]. \end{aligned}$$

Furthermore, we define the rectangular increments of covariance as

$$R\begin{pmatrix} s,t \\ s',t' \end{pmatrix} := \left(\mathbb{E}\left(X_{s,t}^i X_{s',t'}^j\right)\right)_{i,j=1}^d .$$

Using Kolmogorov's continuity and Gaussian hypercontractivity, we obtain the Hölder regularity of process X.

PROPOSITION 1.2. *Assume there exists positive* ϱ *and* M *such that for every* $0 \le s \le t \le T$,

$$\left|R\begin{pmatrix} s,t \\ s,t \end{pmatrix}\right| \le M|t-s|^{1/\varrho}.$$

Then, for every $\alpha < 1/(2\varrho)$ *there exists* $K_\alpha \in L^q$, *for all* $q < \infty$, *such that*

$$|X_{s,t}(\omega)| \le K_\alpha(\omega)|t-s|^\alpha.$$

Proof. Without loss of generality, we set $d = 1$, otherwise, we can consider componentwise. Recall the Kolmogorov's continuity criterion, namely, if there exists $q \ge 2, \beta > 1/q$ s.t.

$$\left\|X_{s,t}\right\|_q \lesssim |t-s|^\beta ,$$

then for all $\alpha \in [0, \beta - 1/q)$, there exists $K_\alpha \in L^q$ such that

$$|X_{s,t}| \le K_\alpha \left|t-s\right|^\alpha , \quad a.s.$$

Hence, we only need to show

$$\left\|X_{s,t}\right\|_q \lesssim |t-s|^{\frac{1}{2\varrho}} , \quad \forall q \ge 2.$$

By Gaussian hypercontractivity, we have

$$\left\|X_{s,t}\right\|_q \lesssim \left\|X_{s,t}\right\|_2 \le= \left|R\begin{pmatrix} s,t \\ s,t \end{pmatrix}\right|^{1/2} \le M^{1/2}|t-s|^{\frac{1}{2\varrho}}, \quad \forall q \ge 2,$$

which completes the proof. □

EXAMPLE 1.3. *A continuous and centered Gaussian process* $(B_t)_{t\geqslant 0}$ *with* $B_0 = 0$ *is called a fractional Brownian motion with Hurst parameter* $H \in (0,1)$ *if it has the covariance*

$$\mathbb{E}\left[B_s B_t\right] = \Gamma(s,t) := \frac{1}{2}\left(s^{2H} + t^{2H} - |t-s|^{2H}\right).$$

By simple computation, we see that

$$\begin{aligned} &\mathbb{E}\left(\left(B_{s,t}^H\right)^2\right) = \mathbb{E}\left(\left(B_t^H\right)^2\right) + \mathbb{E}\left(\left(B_t^H\right)^2\right) - 2\mathbb{E}\left(B_t^H B_s^H\right) \\ =&\frac{1}{2}\left(2t^{2H} + 2s^{2H} - 2t^{2H} - 2s^{2H} + |t-s|^{2H}\right) \lesssim |t-s|^{2H} . \end{aligned}$$

Namely, $\frac{1}{\varrho} = 2H$, *i.e.* $H = \frac{1}{2\varrho}$. *Then we have the following schema:*

1. If $H > \frac{1}{2}$, we can apply Young's theory.

2. If $H = \frac{1}{2}$, it is just Brownian motion, we can apply Itô formula.

3. If $H < \frac{1}{2}$, we cannot apply this anymore, since it is not a semimartingale.

Now we want to construct a reasonable lifted process $\mathbf{X} := (X, \mathbb{X}) \in C_g^\alpha\left([0,T], \mathbb{R}^d\right)$ with suitable α, i.e.

1. Chen's relation:
$$\delta \mathbb{X}_{s,u,t} = X_{s,u} \otimes X_{u,t}. \tag{1.1}$$

2.
$$\operatorname{Sym}\left(\mathbb{X}_{s,t}\right) = \frac{1}{2} X_{s,t} \otimes X_{s,t}. \tag{1.2}$$

3.
$$\|X\|_\alpha := \sup_{s \neq t \in [0,T]} \frac{|X_{s,t}|}{|t-s|^\alpha} < \infty, \quad \|\mathbb{X}\|_{2\alpha} := \sup_{s \neq t \in [0,T]} \frac{|\mathbb{X}_{s,t}|}{|t-s|^{2\alpha}} < \infty. \tag{1.3}$$

The construction of Gaussian rough path is similarly as the one for Brownian motion, namely, we first define the integral
$$\mathbb{X}_{s,t}^{i,j} := \int_s^t X_{s,r}^i \mathrm{d} X_r^j$$
in L^2 sense, and then find a modification. In particular, the only possible choice for this setting should be
$$\mathbb{X}_{s,t}^{i,j} \stackrel{!}{=} \begin{cases} \int_s^t X_{s,r}^i \mathrm{d} X_r^j, & \text{if } 1 \leq i < j \leq d, \\ \frac{1}{2}\left(X_{s,t}^i\right)^2, & \text{if } i = j, \\ -\mathbb{X}_{s,t}^{j,i} + X_{s,t}^i X_{s,t}^j, & \text{if } 1 \leq j < i \leq d \end{cases} \tag{1.4}$$

Note the followings:

1. By (1.4), we only need to consider $\mathbb{X}_{s,t}^{i,j}$ for $1 \leq i < j \leq d$. For the sake of notation we write $\left(X, \tilde{X}\right)$ rather than $\left(X^i, X^j\right)$.

2. It is enough to consider the unit interval, since the interval $[s,t]$ is handled by considering $\left(X_{s+\tau(t-s)} : 0 \leq \tau \leq 1\right)$.

Similar as the case for Brownian motion, we first define the integral in L^2 sense, namely
$$\int_0^1 X_{0,u} \, \mathrm{d}\tilde{X}_u := \lim_{|\mathcal{P}| \downarrow 0} \sum_{[s,t] \in \mathcal{P}} X_{0,\xi} \tilde{X}_{s,t} \quad \text{with } \xi \in [s,t], \tag{1.5}$$

where the limit is understood in probability.

REMARK 1.4. *Assume now $X, \tilde{X}$ are semimartingale. By classic stochastic analysis:*

1. $\xi = s$ ("left-point evaluation") leads to the Itô integral.

2. $\xi = t$("right-point evaluation") to the backward Itô.

3. $\xi = (s+t)/2$ to the Stratonovich integral.

On the other hand, all these integrals only differ by a bracket term $\langle X, \tilde{X}\rangle$ which vanishes if $X, \tilde{X}$ are independent. While we do not assume a semimartingale structure here, we do have the standing assumption of componentwise independence. This suggests a Riemann sum approximation of (1.5) in which we expect the precise point of evaluation to play no role.

For a partition $\mathcal{P}$, we use the following notation:

$$\int_{\mathcal{P}} X_{0,r} \mathrm{d}\tilde{X}_r := \sum_{[s,t]\in\mathcal{P}} X_{0,s}\tilde{X}_{s,t}.$$

In order to show this forms a Cauchy sequence in L^2, the rectangular increments of covariance plays an important role. To this end, we define the following $2D-$variation:

DEFINITION 1.5 (ϱ-Variation)**.** *Let $I, I' \subset \mathbb{R}$ be two intervals. For a function $R : I^2 \times I'^2 \to \mathbb{R}^{d\times d}$, we define its $\varrho-$ variation as*

$$\|R\|_{\varrho;I\times I'} := \left(\sup_{\mathcal{P}\subset I} \sum_{[s,t]\in\mathcal{P}} \sum_{[s',t']\in\mathcal{P}'} \left| R\begin{pmatrix} s,t \\ s',t' \end{pmatrix}\right|^{\varrho}\right)^{\frac{1}{\varrho}}.$$

For this variation, we have the following generalised Young's maximal inequality, namely, if $\|R\|_{\varrho}, \left\|\tilde{R}\right\|_{\varrho'}$ are finite with $\frac{1}{\varrho}+\frac{1}{\varrho'} > 1$, then it holds

$$\left|\sum_{[s,t]\in\mathcal{P},[s',t']\in\mathcal{P}'} R\begin{pmatrix} 0,s \\ 0,s' \end{pmatrix} \tilde{R}\begin{pmatrix} s,t \\ s',t' \end{pmatrix}\right| \lesssim \|R\|_{\varrho} \left\|\tilde{R}\right\|_{\varrho'}.$$

In our case, if we assume $\varrho < 2$, then by the fact $X \perp \tilde{X}$, we have

$$\begin{aligned} &\sup_{\substack{\mathcal{P}\subset I \\ \mathcal{P}'\subset I'}} \left|\mathbb{E}\left(\int_{\mathcal{P}} X_{0,s}\, \mathrm{d}\tilde{X}_s \int_{\mathcal{P}'} X_{0,s}\, \mathrm{d}\tilde{X}_s\right)\right| \\ \overset{indp.}{=} &\sup_{\substack{\mathcal{P}\subset I \\ \mathcal{P}'\subset I'}} \left|\sum_{\substack{[s,t]\in\mathcal{P} \\ [s',t']\in\mathcal{P}'}} R\begin{pmatrix} 0,s \\ 0,s' \end{pmatrix} \tilde{R}\begin{pmatrix} s,t \\ s',t' \end{pmatrix}\right| \lesssim \|R_X\|_{\varrho;[0,1]^2} \left\|\tilde{R}_{\tilde{X}}\right\|_{\varrho;[0,1]^2}. \end{aligned}$$

With some efforts, one can show that

$$\lim_{\varepsilon\to 0} \sup_{\substack{\mathcal{P},\mathcal{P}' \\ |\mathcal{P}|\vee|\mathcal{P}'|<\varepsilon}} \left|\int_{\mathcal{P}} X_{0,r}\mathrm{d}\tilde{X}_r - \int_{\mathcal{P}'} X_{0,r}\mathrm{d}\tilde{X}_r\right|_{L^2} = 0.$$

we can use this to show the L^2 existence of

$$\int_0^1 X_{0,r}\mathrm{d}\tilde{X}_r.$$

And hence, we have the following theorem:

THEOREM 1.6 (Existence of Gaussian Rough Path)**.** *Let $(X_t : 0 \le t \le 1) \in \mathbb{R}^d$ be a Gaussian with $\mathbb{E}(X) \equiv 0$ and $X_i \perp X_j$ for all $i \neq j$. Assume that there exists $\varrho \in [1, 2)$, $M > 0$ such that*

$$\|R_{X^i}\|_{\varrho;[s,t]^2} \le M|t-s|^{1/\varrho}, \quad \forall i, 0 \le s \le t \le 1,$$

then

1. $\mathbb{X}_{s,t}$ *defined as (1.4) exists in* L^2 *sense.*

2. *For any* $\alpha < \frac{1}{2\varrho}$ *with probability one,* $(X, \mathbb{X})$ *satisfies (1.1), (1.2) and (1.3). In particular, for* $\varrho \in \left[1, \frac{3}{2}\right)$ *and any* $\alpha \in \left(\frac{1}{3}, \frac{1}{2\varrho}\right)$ *we have* $(X, \mathbb{X}) \in \mathcal{C}_g^\alpha$.

For fraction Brownian motion, this means $\varrho = \frac{1}{2H} \in \left[1, \frac{3}{2}\right)$, i.e. $H \in \left(\frac{1}{3}, \frac{1}{2}\right]$.

1.2 Fractional Brownian motion

Now we want to check when can we deduce the condition on rectangular increments. To this end, we assume

ASSUMPTION 1.7. $X := (X_1, \ldots, X_d)$ *is a centred continuous Gaussian process with independent components and stationary increments.*

Due to the stationary increments, the law of this process is fully determined by

$$\sigma^2(u) := \mathbf{E}\left[X_{t,t+u}^2\right] = R\begin{pmatrix} t, t+u \\ t, t+u \end{pmatrix}.$$

In order to verify this, one have the following observation:

LEMMA 1.8. *Assume that* $\sigma^2(\cdot)$ *is concave on* $[0, h]$ *for some* $h > 0$. *Then,*

1. $$\mathbf{E}\left[X_{s,t}X_{u,v}\right] \le 0, \quad \forall 0 \le s \le t \le u \le v \le h.$$

2. *If in addition* $\sigma^2(\cdot)$ *is non-decreasing on* $[0, h]$, *then*

$$0 \le \mathbf{E}\left[X_{s,t}X_{u,v}\right] \le \sigma^2(v-u), \quad \forall 0 \le s \le u \le v \le t \le h.$$

This comes directly from the concave property. There is nothing interesting, hence, I will omit the proof. With this in hand, we are able to state a criterion on the $\varrho-$norm of covariance.

THEOREM 1.9. *Let* X *be a real-valued Gaussian process with stationary increments and* $\sigma^2(\cdot)$ *concave and non-decreasing on* $[0, h]$, *some* $h > 0$. *Assume also, for constants* $L, \varrho \ge 1$, *and all* $\tau \in [0, h]$

$$\left|\sigma^2(\tau)\right| \le L|\tau|^{1/\varrho}$$

Then the covariance of X *has finite* ϱ*-variation. More precisely*

$$\|R_X\|_{\varrho;[s,t]^2} \le M|t-s|^{1/\varrho} \tag{1.6}$$

for all intervals $[s, t]$ *with length* $|t - s| \le h$ *and some* $M = M(\varrho, L) > 0$.

Proof. Consider some interval $[s, t]$ with length $|t - s| \le h$ and let $\mathcal{D} = \{t_i\}, \mathcal{D}' = \left\{t'_j\right\}$ be two dissections of $[s, t]$. For fixed t_i, t_{i+1}, we claim

Claim. *It holds*

$$\sum_{t'_j \in \mathcal{D}'} \left|\mathbf{E}\left(X_{t_i,t_{i+1}} X_{t'_j, t'_{j+1}}\right)\right|^\varrho \le L\left|t_{i+1} - t_i\right|.$$

Suppose we have this, then we see that

$$\left(\sum_{t_i\in D}\sum_{t'_j\in\mathcal{D}'}\left|\mathbf{E}\left(X_{t_i,t_{i+1}}X_{t'_j,t'_{j+1}}\right)\right|^{\varrho}\right)^{\frac{1}{\varrho}}\leq L\left|t-s\right|.$$

In order to show the claim note that

$$\begin{aligned}\sum_{t'_j\in\mathcal{D}'}\left|\mathbf{E}\left(X_{t_i,t_{i+1}}X_{t'_j,t'_{j+1}}\right)\right|^{\varrho}&\lesssim\left\|\mathbf{E}X_{t_i,t_{i+1}}X_{\cdot}\right\|^{\varrho}_{\varrho;[s,t]}\\&\lesssim\underbrace{\left\|\mathbf{E}X_{t_i,t_{i+1}}X_{\cdot}\right\|^{\varrho}_{\varrho;[s,t_i]}}_{=:\mathrm{I}}+\underbrace{\left\|\mathbf{E}X_{t_i,t_{i+1}}X_{\cdot}\right\|^{\varrho}_{\varrho;[t_i,t_{i+1}]}}_{=:\mathrm{II}}+\underbrace{\left\|\mathbf{E}X_{t_i,t_{i+1}}X_{\cdot}\right\|^{\varrho}_{\varrho;[t_{i+1},t]}}_{=:\mathrm{III}}.\end{aligned}\tag{1.7}$$

For all three terms we can apply Lemma 1.8 to get the desired bound, for instance for the second term, note that

$$\mathrm{II}=\sup_{\mathcal{D}'}\sum_{t'_j\in\mathcal{D}'}\left|\mathbf{E}X_{t_i,t_{i+1}}X_{t'_j,t'_{j+1}}\right|^{\varrho}\leq\sup_{\mathcal{D}'}\sum_{t'_j\in\mathcal{D}'}\left|\sigma^2\left(t'_{j+1}-t'_j\right)\right|^{\varrho}\leq L\left|t_{i+1}-t_i\right|.$$

□

COROLLARY 1.10 ([FH20, Corollary 10.10]). *Let $X=\left(X^1,\dots,X^d\right)$ be a centred continuous Gaussian process with independent components such that each X^i satisfies the assumption of the Theorem 1.9, with common values of h,L and $\varrho\in[1,3/2)$. Then X, restricted to any interval $[0,T]$, lifts to $\mathbf{X}=(X,\mathbb{X})\in\mathcal{C}^{\alpha}_{g}\left([0,T],\mathbf{R}^d\right)$.*

Proof. Set $I_n=[(n-1)h,nh]$ so that $[0,T]\subset I_1\cup I_2\cup\cdots\cup I_{[T/h]+1}$. On each interval I_n, we may apply Theorem 1.9 to lift $X_n:=X|_{I_n}$ to a (random) rough path $\mathbf{X}_n\in\mathcal{C}^{\alpha}_{g}\left(I_n,\mathbf{R}^d\right)$. The concatenation of $\mathbf{X}_1,\mathbf{X}_2,\dots$ then yields the desired rough path lift on $[0,T]$. □

With this in hand, we are finally to deduce the case for fractional Brownian motion.

EXAMPLE 1.11 (Fractional Brownian motion, [FH20, Example 10.11]). *Clearly, d-dimensional fractional Brownian motion B^H with Hurst parameter $H\in\left(\frac{1}{3},\frac{1}{2}\right]$ satisfies the assumptions of the above theorem / corollary for all components with*

$$\sigma(u)=u^{2H}$$

obviously non-decreasing and concave for $H\leq\frac{1}{2}$ and on any time interval $[0,T]$. This also identifies

$$\varrho=\frac{1}{2H}$$

and $\varrho<\frac{3}{2}$ translates to $H>\frac{1}{3}$ in which case we obtain a canonical geometric rough path $\mathbf{B}^H=\left(B^H,\mathbb{B}^H\right)$ associated to fractional Brownian motion.

1.3 Exponential integrability

Now we want to show a generalised Fernique's theorem for Gaussian rough path. Recall that the original Fernique's theorem is a result about Gaussian measures on Banach spaces. It extends the finite-dimensional result that a Gaussian random variable has exponential tails. Namely, if γ is a Gaussian measure on separable Banach space B, then there exists $\alpha>0$ such that

$$\int_X\exp\left(\alpha\|x\|^2\right)\gamma(\mathrm{d}x)<\infty.$$

Now our goal is to show that under the previous condition, there exists $\eta > 0$ such that

$$\mathbb{E}\left(e^{\eta\|\mathbf{X}\|_\alpha^2}\right) < \infty. \tag{1.8}$$

To this end, we will need Cameron-Martin regularity. Let's first recall the definition of Cameron-Martin space. Let γ be a Gaussian on $(B, \mathcal{B}(B))$, then its dual $B^* \subset L^2(X,\gamma)$, and we can define the continuous inclusion $j: B^* \to L^2(X,\gamma)$ as

$$j(f) := f - a_\gamma(f)$$

with

$$a_\gamma(f) := \int_B f(x)\,\gamma(\mathrm{d}x).$$

Then we define X^*_γ as the closure of $j(B)$ in $L^2(B,\gamma)$. Furthermore, we define $R_\gamma: X^*_\gamma \to (X^*)^*$ as

$$(R_\gamma(f)\,g) = \int_X f(x)\,(g(x) - a_\gamma(x))\,\gamma(\mathrm{d}x).$$

In particular, one can show that $R_\gamma(X^*) \subset X$ in the sense that for all $f \in X^*$, there exists $y_f \in X$ such that

$$R_\gamma(f)\,g = g(y_f), \quad \forall g \in X^*.$$

Then we define the Cameron-Martin space as

$$\mathcal{H}_{\mathrm{CM}} := \left\{h \in X \mid \exists \hat{h} \in X^*_\gamma \textit{ such that } h = R_\gamma\left(\hat{h}\right)\right\}.$$

And we define the norm on it as

$$\|h\|_{\mathcal{H}_{\mathrm{CM}}} := \left\|\hat{h}\right\|_{L^2}.$$

In particular, it forms a Hilbert space with

$$\langle h, g\rangle = \left\langle \hat{h}, \hat{g}\right\rangle.$$

And we call $(B, \mathcal{H}_{\mathrm{CM}}, \gamma)$ as **abstract Wiener space**. In our case, the underlying space is $C\left([0,T];\mathbb{R}^d\right)$ and X is a Gaussian with $X(\omega) = \omega$. Then the Cameron-Martin space $\mathcal{H} \subset C\left([0,T];\mathbb{R}^d\right)$ consists of paths $t \mapsto h_t := \mathbb{E}(ZX_t)$ where

$$Z \in \mathcal{W}^1 := \overline{\mathrm{span}\{X^i_t : t \in [0,T], 1 \le i \le d\}}^{L^2(\gamma)}.$$

The key ingredient to show (1.8) is the following theorem:

THEOREM 1.12 (Generalised Fernique theorem)**.** *Assume $(E, \mathcal{H}, \mu)$ is an abstract Wiener space. Let $a, \sigma \in (0,\infty)$ and consider measurable maps $f, g : E \to [0,\infty]$ such that*

1. $$\mu(\{x : g(x) \le a\}) > 0.$$

2. *There exists a null-set N such that*

$$f(x) \le g(x-h) + \sigma\|h\|_{\mathcal{H}}, \quad \forall x \in N^c,\ h \in \mathcal{H}.$$

Then $f(\cdot)$ has Gaussian tail, more precisely, there exists $\eta > 0$ such that

$$\mathbb{E}\left(\exp\left(\eta\,|f(x)|^2\right)\right)\gamma(\mathrm{d}x) < \infty.$$

Hence, to show (1.8), we just need to do the following:

1. We set $f(\omega) := \|\mathbf{X}(\omega)\|_\alpha$ and show that $\|\mathbf{X}(\omega)\|_\alpha < \infty$ for a.e. ω.
2. And there exists $C, \sigma > 0$ such that

$$\|\mathbf{X}(\omega)\|_\alpha \leq C\left(\|\mathbf{X}(\omega - h)\|_\alpha + \sigma \|h\|_{\mathcal{H}}\right), \quad \forall h \in \mathcal{H}_{\mathrm{CM}}. \tag{1.9}$$

We have already seen a sufficient condition for the first criterion, it turns out it will also implies the second one, and hence we obtain

THEOREM 1.13 ([FH20, Theorem 11.9])**.** *Let $(X_t : 0 \leq t \leq T)$ be a d-dimensional, centred Gaussian process with independent components and covariance R such that there exists $\varrho \in \left[1, \frac{3}{2}\right)$ and $M < \infty$ such that for every $i \in \{1, \ldots, d\}$ and $0 \leq s \leq t \leq T$,*

$$\|R_{X^i}\|_{\varrho-\mathrm{var};[s,t]^2} \leq M|t-s|^{1/\varrho}.$$

Then, for any $\alpha \in \left(\frac{1}{3}, \frac{1}{2\varrho}\right)$, the associated rough path $\mathbf{X} = (X, \mathbb{X}) \in \mathcal{C}_g^\alpha$ built in Theorem 1.6 is such that there exists $\eta = \eta(M, T, \alpha, \varrho)$ with

$$\mathbf{E}\left(\exp\left(\eta \|\mathbf{X}\|_\alpha^2\right)\right) < \infty.$$

Hence, we only need to show (1.9). Instead of working on Hölder space, we will now use the following space:

$$\mathcal{C}^{p\text{-var}}\left([0,T], \mathbf{R}^d\right) := \left\{X \in C\left([0,T]; \mathbb{R}^d\right) \mid \|X\|_{p-\mathrm{var};[0,T]} < \infty\right\},$$

where

$$\|X\|_{p-\mathrm{var};[0,T]} := \left(\sup_{\mathcal{P}} \sum_{[s,t]\in\mathcal{P}} |X_{s,t}|^p\right)^{\frac{1}{p}}. \tag{1.10}$$

with supremum taken over all partitions of $[0,T]$ and this constitutes a seminorm on $\mathcal{C}^{p\text{-var}}$. The 1-variation ($p = 1$) of such a path is of course nothing but its length, possibly $+\infty$.

It has the following connection with Hölder regularity:

PROPOSITION 1.14. *Suppose $f \in C\left([0,T], \mathbb{R}^d\right)$, then:*

1. *If f is $\alpha-$Hölder continuous, then*

$$\|X\|_{p-\mathrm{var};[0,T]} \leq T^\alpha \|X\|_{\alpha;[0,T]}$$

   with $p := \frac{1}{\alpha}$.

2. *Conversely, if f is $p-$variation, then there exists reparameterization such that $f \circ \tau$ is $\frac{1}{p}$ Hölder continuous.*

Instead of using Hölder regularity, we will consider rough path of $p-$variation, and we write $\mathbf{X} := (X, \mathbb{X}) \in \mathcal{C}^{p-var}\left([0,T], \mathbb{R}^d\right)$ if (1.1) and (1.2) holds and

$$\|\mathbb{X}\|_{p/2-\mathrm{var};[0,T]} \stackrel{\text{def}}{=} \left(\sup_{\mathcal{P}} \sum_{[s,t]\in\mathcal{P}} |\mathbb{X}_{s,t}|^{p/2}\right)^{2/p} < \infty$$

(As before, we shall drop $[0,T]$ from our notation whenever the time horizon is fixed.) The homogeneous p-variation rough path norm (over $[0,T]$) is then given by

$$\|\mathbf{X}\|_{p\text{-var };[0,T]} = \|\mathbf{X}\|_{p\text{-var }} := \|X\|_{p\text{-var }} + \sqrt{\|\mathbb{X}\|_{p/2-\mathrm{var}}}.$$

REMARK 1.15. *Originally, we have $\alpha \in \left(\frac{1}{3}, \frac{1}{2}\right]$, now by the relation $p = \frac{1}{\alpha}$, we have $p \in [2,3)$.*

Hence, we change (1.8) to the following

$$\|\mathbf{X}(\omega)\|_{p-var} \leq C\left(\|\mathbf{X}(\omega-h)\|_{p-var} + \sigma\|h\|_{\mathcal{H}}\right), \quad \forall h \in \mathcal{H}_{\mathrm{CM}}. \tag{1.11}$$

Now we see that the $\|\cdot\|_{\mathcal{H}}$ is not convenient. Luckily, we can embed $\mathcal{H}_{\mathrm{CM}}$ into the following space:

PROPOSITION 1.16 ([FV11, Proposition 11.2])**.** *Assume the covariance $R : (s,t) \mapsto \mathbf{E}(X_s \otimes X_t)$ is of finite $\varrho-$ variation (in 2D sense) for $\varrho \in [1,\infty)$. Then $\mathcal{H}$ is continuously embedded in the space of continuous paths of finite ϱ-variation. More, precisely, for all $h \in \mathcal{H}$ and all $s < t$ in $[0,T]$*

$$\|h\|_{\varrho-\mathrm{var};[s,t]} \leq \|h\|_{\mathcal{H}}\sqrt{\|R\|_{\varrho-\mathrm{var};[s,t]^2}}.$$

Proof. Without loss of generality, we assume X, h are scalar. Let $h \in \mathcal{H}$, i.e. $h_t = \mathbb{E}(ZX_t)$ for some $Z \in \mathcal{W}^1$. We may assume without loss of generality (by scaling), that $\|h\|^2_{\mathcal{H}} := \mathbf{E}\left(Z^2\right) = 1$. Let (t_j) be a dissection of $[s,t]$. Let ϱ' be the Hölder conjugate of ϱ. Using duality for ϱ^{ϱ}-spaces, we have [1]

$$\begin{aligned}
\left(\sum_j \left|h_{t_j,t_{j+1}}\right|^{\varrho}\right)^{1/\varrho} &= \sup_{\beta,|\beta|_{l'}\leq 1} \sum_j \left\langle \beta_j, h_{t_j,t_{j+1}}\right\rangle = \sup_{\beta,|\beta|_{l\varrho'}\leq 1} \mathbf{E}\left(Z\sum_j \left\langle \beta_j, X_{t_j,t_{j+1}}\right\rangle\right) \\
&\leq \sup_{\beta,|\beta|_{l\varrho'}\leq 1} \sqrt{\mathbb{E}\left(Z^2\right)}\sqrt{\sum_{j,k}\beta_j\beta_k\mathbb{E}\left(X_{t_j,t_{j+1}}X_{t_k,t_{k+1}}\right)} \\
&\leq \sup_{\beta,|\beta|_{l\varrho'}\leq 1} \sqrt{\left(\sum_{j,k}|\beta_j|^{\varrho'}|\beta_k|^{\varrho'}\right)^{\frac{1}{\varrho'}}\left(\sum_{j,k}\left|\mathbb{E}\left(X_{t_j,t_{j+1}}X_{t_k,t_{k+1}}\right)\right|^{\varrho}\right)^{\frac{1}{\varrho}}} \\
&\leq \left(\sum_{j,k}\left|\mathbf{E}\left(X_{t_j,t_{j+1}}\otimes X_{t_k,t_{k+1}}\right)\right|^{\varrho}\right)^{1/(2\varrho)} \leq \sqrt{\|R\|_{\varrho\text{-var};\,[s,t]^2}}.
\end{aligned}$$

The proof is then completed by taking the supremum over all dissections (t_j) over $[0,t]$. ☐

Now we want to find a relation between $\mathbf{X}(\omega+h)$ and $\mathbf{X}(\omega)$. As an ansatz, we define for a rough path $\mathbf{X} := (X, \mathbb{X})$, we define its translation in direction h as

$$T_h(\mathbf{X}) := \left(X^h, \mathbb{X}^h\right)$$

where $X^h := X + h$ and

$$\begin{aligned}
\mathbb{X}^h_{s,t} :=& \left(\int_s^t X^{h,i}_{s,r}\mathrm{d}X^{h,j}_r\right)^d_{i,j=1} = \left(\int_s^t \left(X^i_{s,r}+h^i_{s,r}\right)\mathrm{d}\left(X^j_{s,r}+h^j_{s,r}\right)\right)^d_{i,j=1} \\
=&\mathbb{X}_{s,t} + \int_s^t h_{s,r}\otimes\mathrm{d}X_r + \int_s^t X_{s,r}\otimes\mathrm{d}h_r + \int_s^t h_{s,r}\otimes\mathrm{d}h_r.
\end{aligned}$$

provided that $h \in \mathcal{C}^{q-var},\ X \in \mathcal{C}^{p-var}$ with

$$\frac{1}{p} + \frac{1}{q} > 1.$$

Now recall that we have for $(X_t)_{t\in[0,1]}$ a $d-$dimensional centered continuous Gaussian process such that $X^i \perp X^j$ and

$$\|R_{X_i}\|_{\varrho;[s,t]} \lesssim |t-s|^{\frac{1}{\varrho}}$$

with $\varrho \in \left[1, \frac{3}{2}\right)$, it holds $\mathbf{X} = (X, \mathbb{X}) \in \mathcal{C}_g^{p-var}$ with $\frac{1}{p} \in \left(\frac{1}{3}, \frac{1}{2\varrho}\right)$ and $\mathcal{H}_{\text{CM}} \hookrightarrow C^{\varrho-var}$. In particular, this implies

$$\frac{1}{p} + \frac{1}{\varrho} > \frac{1}{3} + \frac{2}{3} > 1, \quad \frac{1}{\varrho} + \frac{1}{\varrho} = \frac{4}{3} > 1.$$

Hence, all terms on the right hand side are well-defined. In order to deduce the inequality

$$\|T_h(\mathbf{X})\|_{p-var} \lesssim \left(\|\mathbf{X}\|_{p-var} + \|h\|_{q-var}\right),$$

note that $p > \varrho$, then

$$\left\|X^h\right\|_{p-var} \leq \|X\|_{p-var} + \underbrace{\|h\|_{p-var}}_{\leq \|h\|_{\varrho-var}}.$$

and

$$\max\left\{\left\|\int_0^\cdot h_{s,r} \otimes \mathrm{d}X_r\right\|_{p-var}, \left\|\int_0^\cdot h_{s,r} \otimes \mathrm{d}h_r\right\|_{p-var}, \left\|\int_0^\cdot X_{s,r} \otimes \mathrm{d}h_r\right\|_{p-var}\right\} \lesssim \|h\|_{q-var}\|X\|_{p-var}.$$

Then use the estimate $\sqrt{ab} \leq a+b$ for $a, b \in \mathbb{R}_+$ in view of the homogeneous norm (which involves $\mathbb{X}^h$ with a square root), we can conclude the claim. Now the only thing we need to show is that for all $h \in \mathcal{H}_{\text{CM}}$, it holds

$$T_h(\mathbf{X}(\omega)) = \mathbf{X}(\omega + h), \quad for \ a.e. \ \omega.$$

Suppose we have this, then we have

$$\|\mathbf{X}(\omega)\| = \|T_h(\omega - h)\| \leq C\left(\|\mathbf{X}(\omega - h)\| + \|h\|_{\mathcal{H}}\right).$$

THEOREM 1.17 ([FH20, Theorem 11.5]). *Assume $(X_t : 0 \leq t \leq T)$ is a continuous d-dimensional, centred Gaussian process with independent components and covariance R such that there exists $\varrho \in \left[1, \frac{3}{2}\right)$ and $M < \infty$ such that for every $i \in \{1, \ldots, d\}$ and $0 \leq s \leq t \leq T$,*

$$\|R_{X^i}\|_{\varrho\text{-var};[s,t]^2} \leq M|t-s|^{1/\varrho}$$

Let $\alpha \in \left(\frac{1}{3}, \frac{1}{2\varrho}\right]$ and $\mathbf{X} = (X, \mathbb{X}) \in \mathcal{C}^\alpha\left([0,T], \mathbf{R}^d\right)$ a.s. be the random Gaussian rough path constructed in Theorem 1.9. Then

$$\mathbb{P}\left(\{\omega \mid \mathbf{X}(\omega + h) = T_h(\mathbf{X}(\omega)) \ for \ all \ h \in \mathcal{H}\}\right) = 1.$$

Proof. In order to prove the rest of this theorems, we need to take a close look at the construction of Gaussian rough path. Recall that we use Kolmogorov's criterion to conclude that there exists a modification $\mathbf{X} := (X, \mathbb{X})$ such that for almost every $\omega \in C\left([0,T], \mathbb{R}^d\right)$, $\mathbf{X}(\omega)$ is $\alpha-$Hölder (or $\frac{1}{\alpha}$-variation). Now we define

$$N_1 := \left\{\omega \in \mathcal{C}\left([0,T], \mathbb{R}^d\right) \mid \mathbf{X}(\omega) \ is \ not \ \alpha-\text{Hölder}\right\}.$$

In particular, for any $\omega \in N_1^c$, $h \in \mathcal{H}$, $\omega + h \in N_1^c$. Furthermore, recall that $\mathbb{X}_{s,t}$ was first constructed as an L^2-limit, in particular, there exists a sequence of partitions $(\mathcal{P}^m) \subset [s,t]$ such that

$$\mathbb{X}_{s,t}(\omega) = \lim_{m\to\infty} \int_{\mathcal{P}^m} X \otimes \mathrm{d}X \ exists \ for \ a.e. \ \omega. \tag{1.12}$$

And we denote $N_{2,[s,t]}$ as the set of ω such that (1.12) does not hold. Now we define

$$N_2 := \bigcap_{[s,t]\ dyadic} N_{2,[s,t]}.$$

Now choose $\omega \in (N_1 \cup N_2)^c$ and the aforementioned partition $(\mathcal{P}^m)$ and note that

$$\begin{aligned}
&\int_{\mathcal{P}^m} X(\omega + h) \otimes \mathrm{d}X(\omega + h) \\
&= \underbrace{\int_{\mathcal{P}^m} X(\omega) \otimes \mathrm{d}X(\omega)}_{=:\mathrm{I}} + \underbrace{\int_{\mathcal{P}^m} h \otimes \mathrm{d}X\left(\omega\right) + \int_{\mathcal{P}^m} X\left(\omega\right) \otimes \mathrm{d}h + \int_{\mathcal{P}^m} h \otimes \mathrm{d}h}_{=:\mathrm{II}}.
\end{aligned}$$

Since $\omega \notin N_1$, $X\left(\omega\right)$ and h satisfies the complementary Young regularity, and hence II converges to the respective Young integrals. I converges to $\mathbb{X}_{s,t}\left(\omega\right)$ due to the fact $\omega \notin N_2$. In other words, for all $\omega \in (N_1 \cup N_2)^c$, $h \in \mathcal{H}$ and dyadic time s, t

$$T_h \left(\mathbf{X}\left(\omega\right)\right)_{s,t} = \mathbf{X}\left(\omega\right)_{s,t}.$$

The construction of $\mathbf{X}_{s,t}$ for non-dyadic times was obtained by continuity (see Theorem 1.9) and the above almost-sure identity remains valid. □

References

[FH20] Peter K Friz and Martin Hairer. *A course on rough paths*. Springer, 2020.

[FV11] Peter Friz and Nicolas Victoir. “A note on higher dimensional p-variation”. In: *Electronic Journal of Probability* 16 (2011), pp. 1880–1899.