# Reflected rough differential equations

## Rough paths seminar, 15 July 2022

The content of this seminar is based on the paper [3]. The aim is to develop a solution theory for reflected rough differential equations, which are formally given by

$$\mathrm{d}y_t = f(y_t)\,\mathrm{d}\boldsymbol{X}_t + \mathrm{d}m_t, \quad y_t\,\mathrm{d}m_t = 0. \tag{1}$$

Here $f{:}\,\mathbb{R}\to\mathbb{R}^N$ is regular enough, $\boldsymbol{X}$ is a given $N$-dim. rough path and $(y,m)$ are the unknowns. The novelty, compared to standard RDEs, is that the path $y$ must satisfy the constraint $y_t\in\mathbb{R}_{\geqslant 0}$ for all $t\geqslant 0$, which is enforced by the presence of the *reflection measure* $m$.

## 1 Recap on paths of $p$-variation and controls

The material presented here is mostly from [6], up to changes in terminology and/or notation.

Throughout, $I$ always stands for a real interval, $\mathcal{S}_I := \{(s,t)\in I^2 : s\leqslant t\}$.

**Definition 1. (Control)** *A map $w{:}\,\mathcal{S}_I\to\mathbb{R}_{\geqslant 0}$ is a **control** if $w(s,s)=0$ for all $s\in I$, it satisfies $\lim_{|t-s|\to 0} w(s,t)=0$ and it is superadditive, namely*

$$w(s,u)+w(u,t)\leqslant w(s,t) \quad \forall s\leqslant u\leqslant t.$$

**Definition 2. (Paths of $p$-variation)** *Let $E$ be a Banach space, $p\in[0,\infty)$. A 2-parameter map $h{:}\,\mathcal{S}_I\to E$ is said to be of **finite p-variation**, and we write $g\in\bar{V}_2^p(I;E)$, if*

$$[\![h]\!]_{\bar{V}_2^p(I;E)} := \left(\sup_{\Pi\in\mathcal{P}(I)} \sum_{(t_i,t_{i+1})\in\Pi} |h_{t_i t_{i+1}}|_E^p\right)^{\frac{1}{p}} < \infty.$$

*For $p\in[1,\infty)$, a map $g{:}\,I\to E$ is of finite $p$-variation, $g\in\bar{V}^p(I;E)$, if $\delta g\in\bar{V}_2^p(I;E)$; we set*

$$\|g\|_{\bar{V}^p(I;E)} := \sup_{t\in I}|g_t|_E + [\![\delta g]\!]_{\bar{V}_2^p(I;E)}.$$

*We denote by $V^p(I;E)$ the closure of smooth functions under the $\bar{V}^p$-norm, similarly for $V_2^p(I;E)$.*

**Lemma 3. (cf. Proposition 5.8 from [6])** *The following are equivalent:*

- *i.* $g\in C(I;E)\cap\bar{V}^p(I;E)$;
- *ii.* *there exists a control $w_g$ such that $|\delta g_{st}|\leqslant w_g(s,t)^{\frac{1}{p}}$ for all $(s,t)\in\mathcal{S}_I$.*

*The optimal control $w_g$ is given by $w_g(s,t)=[\![g]\!]_{\bar{V}^p([s,t];E)}^p$.*

Some fundamental facts (assume $E$ finite dimensional if needed):

a) If $g\in C(I;E)\cap\bar{V}^p(I;E)$, then $g\in V^q(I;E)$ for all $q>p$; similarly $\bar{V}^p(I;E)\hookrightarrow\bar{V}^q(I;E)$.

b) Function of bounded 1-variation coincide with the measure theory ones, i.e. $g\in\bar{V}^1(I;E)$ if and only if its derivative $\mathrm{d}g$ in the sense of distributions is a well defined signed Radon measure; thus $\bar{V}^1(I;E)$ contains discontinuous objects (e.g. Heaviside function).

   In particular, if $g{:}\,[\ell_1,\ell_2]\to\mathbb{R}$ is monotone, then $g\in\bar{V}^1$ and $[\![g]\!]_{\bar{V}^1} = |\delta g_{\ell_1,\ell_2}|$.

c) Time-change: $g \in C(I;E) \cap \bar{V}^p(I;E)$ if and only if there exists a continuous increasing function $\tau: I \to [0,1]$ and a path $\gamma \in C([0,1];E)$ which is $1/p-$Hölder such that

$$g_t = \gamma_{\tau(t)}.$$

d) Conversely, if $|\delta g_{st}| \leqslant [\![g]\!]_\alpha |t-s|^\alpha$, then $g \in \bar{V}^{1/\alpha}(I;E)$ and $w_g$ is just $[\![g]\!]_\alpha^{1/\alpha}|t-s|$.

e) If $g \in \bar{V}^p(I;E)$ and $f \in C^\alpha(E;F)$ for $\alpha \in (0,1]$, then $f \circ g \in \bar{V}^{p/\alpha}(I;E)$.

f) If $w_1$ and $w_2$ are controls, then so is $\lambda w_1 + w_2$; for any $\theta \in (0,1)$, so is $w_1(s,t)^\theta w_2(s,t)^{1-\theta}$.

g) If $w$ is a control, then so is $w^p$ for any $p \geqslant 1$.

**Lemma 4.** **(Sewing Lemma with controls)** *Let $\xi > 1$ and let $A: \mathcal{S}_I \to E$ satisfy*

$$|\delta A_{sut}| \leqslant w(s,t)^\xi$$

*for some control $w$. Then there exists $C_\xi > 0$ and a (unique up to constant) map $g: I \to E$ such that*

$$|\delta g_{st} - A_{st}|_E \leqslant C_\xi \, w(s,t)^\xi \quad \forall s \leqslant t.$$

**Basic example:** if $f \in V^p(I;\mathbb{R})$, $g \in \bar{V}^q(I;\mathbb{R})$ with $1/p + 1/q > 1$, then one can define the Young integral $\int f \mathrm{d}g$ as the sewing of $A_{st} := f_s g_{st}$ (in line with the Hölder case for $\alpha = 1/p$, $\beta = 1/q$).

**Definition 5.** *Let $p \in [2,3)$, $N \in \mathbb{N}$; $\boldsymbol{X} = (X, \mathbb{X}) \in \bar{V}^p([0,T];\mathbb{R}^N) \times \bar{V}_2^{p/2}([0,T];\mathbb{R}^{N\times N})$ is a $\mathbb{R}^N$-valued **p-variation rough path** if it satisfies the relation*

$$\delta \mathbb{X}^{ij}_{sut} = \delta X^i_{su} \, \delta X^j_{ut}.$$

*It is a continuous geometric rough path, $\boldsymbol{X} \in V^p([0,T];\mathbb{R}^N) \times V_2^{p/2}([0,T];\mathbb{R}^{N\times N})$ if it can be obtained, in the $p$-variation topology, as the limit of smooth rough paths (with canonical lift).*

**Remark 6.** Condition $\boldsymbol{X} \in \bar{V}^p([0,T];\mathbb{R}^N) \times \bar{V}_2^{p/2}([0,T];\mathbb{R}^{N\times N})$ can be expressed compactly by imposing the existence of a control $w_{\mathbb{X}}$ such that

$$|X_{st}| + |\mathbb{X}_{st}|^{1/2} \leqslant w_{\mathbb{X}}(s,t)^{1/p} \quad \forall (s,t) \in \mathcal{S}_{[0,T]}. \tag{2}$$

Many concepts and results seen until now (controlled RPs, integration against RPs, etc.) naturally extend to $p$-variation setting, thanks to Lemma 4 and properties of controls.

## 2 Skorokhod problem and regularity of Skorokhod map

If $y$ solves (1) and we set $g_t := y_0 + \int_0^t f(y_s) \mathrm{d}\boldsymbol{X}_s$, then we obtain a decomposition $y = g + m$, where $y$ must stay positive and satisfy certain constraints w.r.t. $m$. This is the setting of the so called Skorokhod problem.

**Definition 7.** **(Skorokhod problem)** *Let $I = [\ell_1, \ell_2]$, $g \in C(I;\mathbb{R})$ with $g_{\ell_1} \geqslant 0$. The **Skorokhod problem** in the domain $\mathbb{R}_{\geqslant 0}$ associated with $g$ consists in finding a pair $(y,m) \in C(I;\mathbb{R})^2$ such that:*

*i.* $y_t \geqslant 0$ *for all* $t \in I$;

*ii.* $m_{\ell_1} = 0$, *$m$ is increasing and* $\int_{\ell_1}^{\ell_2} y_t \, \mathrm{d}m_t = 0$;

*iii.* $y_t = g_t + m_t$.

**Remark 8.** Since $m$ is continuous and increasing, $\mathrm{d}m$ is a nonnegative, locally finite measure, thus the integral appearing in *ii.* is meaningful. The three conditions in *ii.* can be equivalently formulated in a compact way as

$$m_t = \int_{\ell_1}^{t} \mathbb{1}_{\{y_u=0\}} |\mathrm{d}m|_u \quad \forall t \in I.$$

**Theorem 9. (Skorokhod)** *There exists a unique solution to the Skorokhod problem associated to $g$ as in Definition 7, which is given by*

$$m_t = \sup_{s\in[\ell_1,t]} g_s^-, \quad y_t = g_t + m_t,$$

*where $x^- = \max\{-x, 0\}$ denotes the negative part. The function $\Phi: g \mapsto m$ is referred to as the* ***Skorokhod map****.*

**Naive idea to tackle (1):** if $y$ solves a reflected equation of the form (1), then it should be a fixed point for the map obtained by composition of the rough integral and the Skorokhod's map

$$y \mapsto \int_0^{\cdot} f(y_t)\mathrm{d}\boldsymbol{X}_t \mapsto \tilde{y}_\cdot := \int_0^{\cdot} f(y_t)\mathrm{d}\boldsymbol{X}_t + \Phi\left(\int_0^{\cdot} f(y_t)\mathrm{d}\boldsymbol{X}_t\right). \tag{3}$$

**Problem:** find a Banach space where the map defined in this way is a contraction. This is actually hard, as the map $\Phi$ doesn't need to be continuous in general; this is a problem already at the *Young* level, not specific to the rough setting.

**Lemma 10. (Lemma 3.5 and Remark 3.6 from [5])** *Let $\alpha \in (0,1)$. Then there exists $C>0$ such that for any $g \in C^\alpha(I)$ and any $[s,t] \subset I$ it holds*

$$\|\Phi(g)\|_{\alpha;[s,t]} \leqslant C\|g\|_{\alpha;[s,t]}.$$

*However, for any $M>0$ one can find examples of $g^1, g^2 \in C^\alpha$ such that*

$$\|g^i\|_\alpha \leqslant 1, \quad \|g^1 - g^2\|_\alpha \leqslant 1, \quad \|\Phi(g^1) - \Phi(g^2)\|_\alpha \geqslant M.$$

The issue is absent in $C(I;\mathbb{R})$ with supremum norms: a straightforward computation yields

$$\sup_{t\in I} |\Phi(g^1)_t - \Phi(g^2)_t| \leqslant \sup_{t\in I} |g_t^1 - g_t^2| \tag{4}$$

The situation seems more optimistic in the framework of $p$-variation norms:

**Theorem 11. (Theorem 2.1 from [4])** *For any $p \in [1,\infty)$ and any $[s,t] \subset I$ it holds*

$$\|\Phi(g^1) - \Phi(g^2)\|_{\bar{V}^p(I;\mathbb{R})} \leqslant \|g^1 - g^2\|_{\bar{V}^p(I;\mathbb{R})}.$$

Indeed, in the Young case (i.e. $X \in V^p([0,T];\mathbb{R}^N)$ with $p \in [1,2)$), Theorem 11 allows to establish existence and uniqueness of solutions to the reflected RDE, cf. Theorem 3.2 from [4].

Unfortunately, in the regime $p \geqslant 2$ this is not enough. Indeed, to set up contractivity for the map (3), we also need contractivity for objects morally of the form $\int f(\Phi(g)_s)\mathrm{d}\mathbb{X}_s$.

Since $\Phi(g) \in V^1(I;\mathbb{R})$, the integral is in fact well-defined, but we cannot control it purely by the $V^p$-norm of $\Phi(g)$, rather we need its $V^q$-norm for $1/q > 1 - 1/p$. However, $\Phi$ is not even Hölder continuous in such target space!

**Lemma 12.** **(Proposition 8 from [3])** *For all $p > q \geqslant 1$ and any $\alpha \in (0, 1]$, the Skorokhod map $\Phi$ is not $\alpha$-Hölder continuous from $\bar{V}^p(I; \mathbb{R})$ to $\bar{V}^q(I; \mathbb{R})$.*

# 3 Main results from [3] and further literature

**Definition 13.** *Let $T > 0$, $a \geqslant 0$, a differentiable function $f\colon \mathbb{R} \to \mathbb{R}^N$ and a $p$-variation $\mathbb{R}^N$-valued rough path $\mathbb{X}$ with $p \in [2, 3)$ be given. A pair $(y, m) \in V^p([0, T]; \mathbb{R}_{\geqslant 0}) \times V^1([0, T]; \mathbb{R}_{\geqslant 0})$ is said to be a **solution the reflected RDE** (1) on $[0, T]$ with initial condition $a$ if there exists a 2-index map $y^\natural \in V_2^{p/3}([0, T]; \mathbb{R})$ such that*

$$\begin{cases} \delta y_{st} = f_i(y_s)\, \delta X_{st}^i + f_{ij}^2(y_s) \mathbb{X}_{st}^{ij} + \delta m_{st} + y_{st}^\natural \\ y_0 = a, \quad m_t = \int_0^t \mathbb{1}_{y_u = 0}\, |\mathrm{d}m|_u, \end{cases} \tag{5}$$

*where $f_{ij}^2(x) := f_i(x) f_j'(x)$.*

**Remark 14.** As in standard rough paths theory, eq. (5) should be interpreted as a local expansion of the function $y$, so that for $|t - s| \ll 1$ it holds

$$y_t \approx y_s + f_i(y_s)\, \delta X_{st}^i + f_{2,ij}(y_s) \mathbb{X}_{st}^{ij} + \delta m_{st}$$

up to higher terms of order $w(s, t)^{p/3}$, where $w$ is a control and $p/3 < 1$; observe that since $m \in V^1$, we don't need to expand the term $\delta m_{st}$ any further. Condition (5) implies that $y$ is controlled by $X$ with "derivative" $y' = f(y)$ (which is actually the original ansatz needed to derive (5) from (1)).

**Theorem 15.** **(Theorem 4 from [3])** *Let $T > 0$, $a > 0$, $f \in C_b^3(\mathbb{R}; \mathbb{R}^N)$ and $(X, \mathbb{X})$ be a continuous geometric $\mathbb{R}^N$-valued rough path of finite $p$-variation for some $p \in [2, 3)$. Then there exists a unique solution $(y, m)$ to problem (5).*

Under weaker assumptions, it is still possible to show existence of solutions.

**Theorem 16.** **(Theorem 12 from [3])** *Let $T > 0$, $a > 0$, $f \in C_b^2(\mathbb{R}; \mathbb{R}^N)$ and $(X, \mathbb{X})$ be a continuous geometric $\mathbb{R}^N$-valued rough path of finite $p$-variation for some $p \in [2, 3)$. Then there exists at least one solution $(y, m)$ to problem (5).*

The proof technique of Theorem 16 easily extends to the higher dimensional setting, namely to the case of $\mathbb{R}^d$-valued paths $y$ which are constrained to stay inside a suitable connected domain $D \subset \mathbb{R}^d$, with reflection measure $m$ active only whenever $y$ reaches the boundary $\partial D$; see Theorem 14 from [3] for this extension.

The same is not true for Theorem 15, whose proof crucially exploits the 1-dimensionality of $y$. It turns out that this is not just a limitation of the proof:

**Theorem 17.** **(Theorem from [7])** *For any $d \geqslant 2$, $N \geqslant 2$ and $p > 2$, one can find a smooth domain $D \subset \mathbb{R}^d$, a smooth function $f \in C^\infty(\mathbb{R}^d; \mathbb{R}^{d \times N})$, a geometric rough path $(X, \mathbb{X})$ of finite $p$-variation and $y_0 \in D$, such that uniqueness of the associated $d$-dimensional Skorokhod problem doesn't hold.*

The counterexample constructed in [7] is actually fairly simple: it is enough to consider $d = N = 2$, $D = \mathbb{R}_{\geqslant 0} \times \mathbb{R}$, $f$ linear. The system in consideration is

$$\begin{cases} \mathrm{d}y = A y \mathrm{d}X - e_1\, \mathrm{d}\gamma + e_1\, \mathrm{d}K \\ y \cdot e_1 \geqslant 0, \quad \mathrm{d}K = \mathbb{1}_{\{y \cdot e_1 = 0\}} |\mathrm{d}K| \end{cases} \quad \text{where} \quad e_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}; \tag{6}$$

here $X$ and $\gamma$ are continuous scalar functions, with nondecreasing $\gamma$.

Due to the simple structure of (6), one can always make sense of it by means of Duhamel's formula, without needing to specify a rough lift for $X$. Theorem 2.5 from [7] shows that, if $X$ is sampled as a fBm of parameter $H<1/2$, then there exists a deterministic $\gamma^H$ such that $\mathbb{P}$-a.s. the problem (6) admits infinitely many solutions starting from $y_0=0$, which are moreover adapted. Same result is not true in the Brownian case, $H=1/2$!

In a different direction: the uniqueness result from Theorem 15 has been further extended in [1], where the proof is also simplified.

# 4 Ideas of proofs

Since there is no clear way to set up a contraction, existence and uniqueness are treated as separate problem, by another standard meta-strategy:

1. Existence by a priori estimates, compactness and then passing to the limit.

2. Uniqueness by direct comparison of two solutions, usually by means of Grönwall lemma.

Interesting aspects involving the relation between RDEs and SDEs in the Brownian setting:

a) In the SDE setting, Step 1. is the most challenging; here compactness means tightness, one most move to another prob. space by Skorokhod's thm, passage to the limit is not obvious.

   Instead Step 2. is relatively simple, thanks to property (4) and standard estimates for stochastic integrals (e.g. Doob, Burkholder); as a classical paper, see [8].

b) In the RDE setting, since the analysis is pathwise, existence is relatively simple.

   However we cannot rely on (4) and the map $\Phi$ doesn't enjoy nice properties in Hölder or $p$-var. spaces (cf. Lemmas 10 and 12), which makes the uniqueness step harder.

## 4.1 Existence

The next lemma provides the main a priori estimate needed.

**Lemma 18. (cf. proof of Theorem 12 in [3])** *Let $(y,m)$ be a solution to (5), then there exist constants $C,L$, depending on $p$, $\|f\|_{C_b^2}$ and $w_{\mathbb{X}}$, such that*

$$|\delta y_{st}|+|\delta m_{st}|\leqslant C w_{\mathbb{X}}(s,t)^{1/p}\qquad \text{for all } s\leqslant t \text{ such that } w_{\mathbb{X}}(s,t)\leqslant L.$$

**Proof.** Since $y\in V^p$, $m\in V^1$, $y^\natural\in V^{p/3}$ and $(X,\mathbb{X})\in V^p\times V^{p/2}$, we can find controls such that

$$|\delta y_{st}|\leqslant w_y(s,t)^{1/p},\quad |\delta m_{st}|\leqslant w_m(s,t),\quad |y^\natural_{st}|\leqslant w_\natural(s,t)^{3/p}$$

and $w_{\mathbb{X}}$ satisfying (2). Then from (5) and basic manipulations, we get

$$w_y^{1/p}(s,t)\lesssim w_{\mathbb{X}}^{1/p}(s,t)+w_{\mathbb{X}}^{2/p}(s,t)+w_m(s,t)+w_\natural^{3/p}(s,t)\quad \forall s\leqslant t. \tag{7}$$

From now on, we will always work with $|t-s|$ small enough so that $w_{\mathbb{X}}(s,t)\leqslant 1$. In order to control $y^\natural$, we compute $\delta y^\natural$ in order to apply Lemma 4:

$$\begin{aligned}
\delta y^\natural_{sut} &= \delta[f(y)]_{su}\cdot\delta X_{ut}+f^2(y_s){:}\,\mathbb{X}_{su}+f^2(y_u){:}\,\mathbb{X}_{ut}-f^2(y_s){:}\,\mathbb{X}_{st}\\
&= (\delta[f(y)]_{su}-f(y_s)\cdot\delta X_{su}\,f'(y_s))\cdot\delta X_{ut}+\delta[f^2(y)]_{su}{:}\,\mathbb{X}_{ut}\\
&= (\delta y_{su}-f(y_s)\cdot\delta X_{su})[\![f'(y)]\!]_{su}\cdot\delta X_{ut}+(f(y_s)\cdot\delta X_{su})([\![f'(y)]\!]_{su}-f'(y_s))\cdot\delta X_{ut}\\
&\quad +\delta[f^2(y)]_{su}{:}\,\mathbb{X}_{ut}
\end{aligned}$$

where we used the notation $[\![f'(y)]\!]_{su} = \int_0^1 f'(y_s + \lambda \delta y_{su})\,\mathrm{d}\lambda$ in agreement with Taylor expansion. Plugging in again the expansion (5) gives

$$|\delta y^\natural| \lesssim \big(w_{\mathbb{X}}^{2/p} + w_m + w_\natural^{3/p}\big) w_{\mathbb{X}}^{1/p} + w_y^{1/p}\, w_{\mathbb{X}}^{2/p} + w_y^{1/p}\, w_{\mathbb{X}}^{2/p};$$

Plugging in again (7), manipulating the controls and sewing overall yields

$$w_\natural(s,t) \leqslant C_{f,p}\big[w_{\mathbb{X}}(s,t) + w_{\mathbb{X}}^{1/3}(s,t)\, w_m^{p/3}(s,t) + w_{\mathbb{X}}^{1/3}\, w_\natural(s,t)\big].$$

Reestricting ourselves to intervals $I$ such that $C_{f,p} w_{\mathbb{X}}(I)^{1/3} \leqslant 1/2$ then gives

$$w_\natural(s,t) \lesssim w_{\mathbb{X}}(s,t) + w_{\mathbb{X}}^{1/3}(s,t)\, w_m^{p/3}(s,t) \quad \forall [s,t] \subset I. \tag{8}$$

To control $w_m$, first observe that $w_m(s,t) = \delta m_{st}$ and $m = \Phi(g)$ for some path $g$, so that

$$w_m(s,t) \leqslant \|g\|_{0,[s,t]} := \sup_{[u,v]\subset[s,t]} |\delta g_{uv}|;$$

by (5), $\delta g_{st} = f(y_s) \cdot \delta X_{st} + f^2(y_s) \colon \mathbb{X}_{st} + y_{st}^\natural$, therefore

$$w_m(s,t) \lesssim w_{\mathbb{X}}(s,t)^{1/p} + w_\natural^{3/p}(s,t). \tag{9}$$

Inserting (8) in (9), we obtain

$$w_m(s,t) \leqslant \tilde{C}_{f,p}\big[w_{\mathbb{X}}(s,t)^{1/p} + w_{\mathbb{X}}^{1/p}(s,t)\, w_m(s,t)\big].$$

Overall, working on intervals $I$ such that

$$w_{\mathbb{X}}(I) \leqslant 1 \wedge (2 C_{f,p})^{-3} \wedge (2\,\tilde{C}_{f,p})^{-p}$$

yields $w_m(s,t) \lesssim w_{\mathbb{X}}(s,t)^{1/p}$, which combined with (7)-(8) gives the conclusion. □

**N.B:** Lemma 18 doesn't require $\boldsymbol{X}$ to be geometric! However, in order to rigorously obtain existence, a compactness argument must be developed, which roughly amounts to:

1. Consider approximate solutions $(y^\varepsilon, m^\varepsilon)$ to the problem associated to smooth $X^\varepsilon$ (for which wellposedness holds classically), infer a (uniform in $\varepsilon$) estimate of the form (18).

2. By compactness, extract a convergent subsequence $(y^\varepsilon, m^\varepsilon) \to (y, m)$ in suitable topology.

3. Show that $(y, m)$ is a solution to the reflected RDE (5). Here property $(X^\varepsilon, \mathbb{X}^\varepsilon) \to (X, \mathbb{X})$ is crucial and this is why Theorems 15-16 are restricted to geometric RPs.

## 4.2 Uniqueness

In [3], following a similar scheme to [2], uniqueness is accomplished by a **doubling of variables** (or tensorization) argument, jointly with the following **Rough Grönwall Lemma**:

**Lemma 19.** **(Lemma 2 from [3])** *Let $T > 0$, $g\colon [0,T] \to \mathbb{R}_{\geqslant 0}$ be a path satisfying*

$$\delta g_{st} \leqslant C\Big(\sup_{r\in[0,t]} g_r\Big) w_1(s,t)^\alpha + w_2(s,t) \quad \forall (s,t) \in \mathcal{S}_{[0,T]} \text{ s.t. } w_1(s,t) \leqslant L$$

*for some $C, L > 0$ and some controls $w_1, w_2$. Then there exists a constant $K = K(\alpha, L)$ such that*

$$\sup_{t \in [0,T]} g_t \leqslant 2 e^{K w_1(0,T)} \left( g_0 + \sup_{t \in [0,T]} e^{-K w_1(0,t)} w_2(0,t) \right).$$

**Some ideas in the proof of Theorem 15.** The "bulk" of the proof is relatively simple: given two solutions $(y, \mu)$, $(z, \nu)$, we want to show that $y \equiv z$, equivalently $|y_t - z_t| = 0$ for all $t \in [0, T]$.

In the classical ODE setting, this can be accomplished by a direct differentiation:

$$|y_t - z_t| = \int_0^t \operatorname{sign}(y_s - z_s)(f(y_s) - f(z_s)) \cdot \mathrm{d}X_s + \int_0^t \operatorname{sign}(y_s - z_s)(\mathrm{d}\mu_s - \mathrm{d}\nu_s). \tag{10}$$

By contradiction, assume uniqueness fails and $|y_t - z_t| \neq 0$ in $[0, \delta]$; then it is not hard to see that the last integral in (10), due to the properties of reflected RDEs, is always decreasing; in fact

$$h_t := \int_0^t \operatorname{sign}(y_s - z_s)(\mathrm{d}\mu_s - \mathrm{d}\nu_s) = -\mu([0,t]) - \nu([0,t]) + \int_0^t \mathbb{1}_{\{y_u = z_u\}} \mathrm{d}(\mu_u + \nu_u).$$

The brutal estimate $h \leqslant 0$ inserted in (10) then yields

$$|y_t - z_t| \leqslant \| f \|_{\mathrm{Lip}} \int_0^t |y_s - z_s| \, |\dot{X}_s| \, \mathrm{d}s$$

which by classical Grönwall immediately yields a contradiction.

**Problem:** can we obtain an analogue of (10) in the RDE setting? The answer is positive.

To prove it, the authors in [3] roughly proceed as follows:

1. For $\varphi$ smooth, derive a "differential" formula for $\varphi(y - z)$; to this end, it is useful to consider the pair $Y = (y, z)$ as solving a 2-dim. RDE of the form

$$\delta Y_{st} = F_i(Y_s) \, \delta X_{st}^i + F_{ij}^2(Y_s) \, \mathbb{X}_{st} + \delta M_{st} + Y_{st}^\natural$$

   for $F_i(Y) = (f_i(y), f_i(z))$, $F_{ij}^2(Y) = (f_{ij}^2(y), f_{ij}^2(z))$, $M = (\mu, \nu)$. This is a **doubling variable** (or **tensorization**) technique. Setting $h(Y) = \varphi(y - z)$, it satisfies an expansion

$$\delta h(Y)_{st} = H_i(Y_s) \delta X_{st}^i + H_{ij}^2(Y_s) \mathbb{X}_{st}^{ij} + \int_s^t \varphi'(y_u - z_u)(\mathrm{d}\mu_u - \mathrm{d}\nu_u) + h_{st}^\natural$$

   for suitable fields $H_i$, $H_{ij}^2$. After several computations, one can find an estimate for $h^\natural$, where the function $\varphi$ enters only through the quantity

$$|||\varphi||| = \sup_{x \in \mathbb{R}} \{ |\varphi'(x)| + |x| |\varphi''(x)| + |x|^2 |\varphi'''(x)| \}.$$

2. Consider a suitable smooth approximation of $\varphi(x) = |x|$, given by $\varphi_\varepsilon(x) = \sqrt{\varepsilon^2 + |x|^2}$; it is not hard to check that $|||\varphi_\varepsilon|||$ is uniformly bounded. In fact one can pass to the limit $\varepsilon \to 0$ and show rigorously an expansion for $\Phi(Y) = |y - z|$ of the form

$$\begin{aligned} \delta \Phi(Y)_{st} \;=\; & \operatorname{sgn}(y_s - z_s) \, (f(y_s) - f(z_s)) \cdot \delta X_{st} + \operatorname{sgn}(y_s - z_s) \, (f^2(y_s) - f^2(z_s)) : \mathbb{X}_{st} \\ & + \int_0^t \operatorname{sign}(y_s - z_s)(\mathrm{d}\mu_s - \mathrm{d}\nu_s) + \Phi_{st}^\natural, \end{aligned}$$

   together with an estimate for $\Phi^\natural$, depending on $y - z$ and $y^\natural - z^\natural$.

3. Use the previous step to derive an estimate for $y^\natural - z^\natural$ in function of $y - z$, as well as a recursive estimate for the latter; finally use Lemma 19 to conclude that $y - z \equiv 0$. $\square$

**Alternative idea of proof from [1], Theorem 4.1.** Recall that in the rough case, the main obstacle in order to set up a more classical contraction procedure was Lemma 12; although the lemma is true on the whole $\bar{V}^p$, there are notable exceptions.

**Lemma 20. (Lemma 2.5 from [1])** *Let $g: [0,T] \to \mathbb{R}$ be a monotone path, then it holds*

$$\|X\|_{\bar{V}^p([0,T];\mathbb{R})} = \|X\|_{\bar{V}^1([0,T];\mathbb{R})} \quad \forall p \in [1,\infty).$$

Now suppose we are given two solutions $(y,\mu)$ and $(z,\nu)$ and that there exists $a \in [0,T)$ such that $y_a \neq z_a$; define $u = \sup\{s \in [0,a): y_s = z_s\}$, so that by continuity $u < a$, $y_u = z_u$ and $y_s \neq z_s$ on $(u,a]$. From now on we will consider the reflected RDE only on $[u,a]$.

Overall, up to shifting/rescaling we may assume $u = 0$, $a = T$ and $y_s \neq z_s$ on $(0,T]$. By one dimensionality and continuity of $(y,z)$, this implies that either $y_s > z_s \geqslant 0$ for all $s > 0$, or $z_s > y_s \geqslant 0$; assume the first case, the second one being identical. But then this implies that $\mu \equiv 0$ and so that the process $\mu - \nu = -\nu$ is monotone!

Applying Lemma 20, combined with Theorem 11, this allows to set up a contraction procedure:

$$\begin{aligned} \|\mu - \nu\|_{V^1} = \|\mu - \nu\|_{V^p} &= \left\| \Phi\left( \int_0^\cdot f(y_r)\mathrm{d}\boldsymbol{X}_r \right) - \Phi\left( \int_0^\cdot f(z_r)\mathrm{d}\boldsymbol{X}_r \right) \right\|_{V^p} \\ &\lesssim \left\| \int_0^\cdot [f(y_r) - f(z_r)]\mathrm{d}\boldsymbol{X}_r \right\|_{V^p} \end{aligned}$$

and from here one can use classical results on stability of rough integrals $y \mapsto \int_0^\cdot f(y_r)\mathrm{d}\boldsymbol{X}_r$, available for $f \in C_b^3$, to set up a closed estimate implying that $y = z$ on some $[0,\delta]$ with $\delta > 0$, yielding a contradiction. $\square$

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