Operations on Contolled Rough Paths

Goal: understand basic operations needed for studying RDE and in particular an Ito's formula for functions of a controlled rough paths:

$$\mathrm{d}Y_t = Y_t'\mathrm{d}\boldsymbol{X}_t + \mathrm{d}\Gamma_t \qquad \left(Y_{0,t} = \int_0^t Y_s'\mathrm{d}\boldsymbol{X}_s + \Gamma_{0,t}\right).$$

for regular enough drift Γ_t. Plan:

(i) - Controlled rough paths as rough paths

(ii) - Composition with regular functions

(iii) - Ito's lemma

Disclaimer: We fix an interval $[0,T]$ and write $\mathcal{C}^\alpha(V) \equiv \mathcal{C}^\alpha([0,T];V)$. Often, we will not specify the Banach space and adridge the notation to $\mathcal{C}^\alpha$.

Preliminary:

- RI against rough path $\int Y\mathrm{d}\boldsymbol{X}$ for $\boldsymbol{X} \in \mathcal{C}^\alpha(V)$, $(Y, Y') \in \mathcal{D}_X^{2\alpha}(\mathcal{L}(V;W))$:

  $\Xi_{u,v} = Y_u X_{u,v} + Y_u' \mathbb{X}_{u,v}$, $\|\delta\Xi\|_{3\alpha} < \infty$

- RI against controlled rough path $\int Y\mathrm{d}Z$ for $\boldsymbol{X} \in \mathcal{C}^\alpha(V)$, $(Y, Y') \in \mathcal{D}_X^{2\alpha}(\mathcal{L}(\bar{W};W))$, $(Z, Z') \in \mathcal{D}_X^{2\alpha}(W)$

  $\Xi_{u,v} = Y_u Z_{u,v} + Y_u' Z_u' \mathbb{X}_{u,v}$, $\|\delta\Xi\|_{3\alpha} < \infty$

1 Controlled RP as RP

Rough integration allows us to compute $\int_s^t X_{s,r} \otimes \mathrm{d}\boldsymbol{X}_r = \mathbb{X}_{s,t}$ (sanity check). This holds because

$$\Xi_{u,v}^s = X_{s,u} \otimes X_{u,v} + \mathbb{X}_{u,v}$$

so that identity holds for any partition. We have relation $(X, \mathrm{Id}) \in \mathcal{D}^{2\alpha}$ and $(X, \mathbb{X}) \in \mathcal{C}^\alpha$. More generally one expects injection

$$\mathcal{D}^{2\alpha}(W) \hookrightarrow \mathcal{C}^\alpha(W)$$

That is, to associate a RP to any controlled RP

$$(Y, Y') \mapsto (Y, \mathbb{Y})$$

$$\mathbb{Y}_{s,t} := \int_s^t Y_{s,r} \otimes \mathrm{d}Y_r = \mathcal{I}(\Xi^s)_{s,t} \qquad \Xi_{u,v}^s = Y_{s,u} \otimes Y_{u,v} + (Y_u' \otimes Y_u')\, \mathbb{X}_{u,v}$$

We have $\|\mathbb{Y}\|_{2\alpha} < \infty$ consequence of $|\mathcal{I}(\Xi^s)_{s,t} - \Xi_{s,t}^s| \lesssim |t-s|^{3\alpha}$ and $\|\Xi_{s,t}^s\|_{2\alpha} < \infty$. Chen's relation is obvious from the fact that $\delta(\int_s^t Y_r \otimes \mathrm{d}Y_r)_{s,u,t} = 0$ (abstract integration property) and that $\delta(\int_s^t Y_s \otimes \mathrm{d}Y_r)_{s,u,t} = Y_{s,u} \otimes Y_{u,t}$.

Lemma 1. *(Consistency).* $\boldsymbol{X}\in\mathcal{C}^\alpha$, $(Y,Y')\in\mathcal{D}_X^{2\alpha}$, $\boldsymbol{Y}\in\mathcal{C}^\alpha$ *(canonical). If* $(\tilde{Z},\tilde{Z}')\in\mathcal{D}_Y^{2\alpha}$, *then setting* $Z_t:=\tilde{Z}$ *and* $Z_t':=\tilde{Z}_t'Y_t'$ $(Z,Z')\in\mathcal{D}_X^{2\alpha}$ *and consistency*

$$\int_s^t \tilde{Z}_r\mathrm{d}\boldsymbol{Y}_r=\int_s^t Z_r\mathrm{d}Y_r$$

Proof. First show that $(Z,Z')\in\mathcal{D}_X^{2\alpha}$:

$$\begin{aligned} Z_{s,t}=\tilde{Z}_{s,t} &= \tilde{Z}_s'Y_{s,t}+O(|t-s|^{2\alpha})\\ &= \tilde{Z}_s'Y_s'X_{s,t}+O(|t-s|^{2\alpha}). \end{aligned}$$

The second integral has local approximation:

$$\begin{aligned} \Xi_{u,v} &= Z_uY_{u,v}+Z_u'Y_u'\mathbb{X}_{u,v}\\ &= Z_uY_{u,v}+\tilde{Z}_uY_u'Y_u'\mathbb{X}_{u,v} \end{aligned}$$

By definition of $\mathbb{Y}$, we have the local approximation estimate $|\mathbb{Y}_{s,t}-(Y_u'\otimes Y_u')\,\mathbb{X}_{u,v}|\sim O(|t-s|^{3\alpha})$ so that the first integral has local approximation

$$\begin{aligned} \tilde{\Xi}_{u,v} &= \tilde{Z}_uY_{u,v}+\tilde{Z}_u'\mathbb{Y}_{u,v}\\ &= \Xi_{u,v}+O(|t-s|^{3\alpha}) \end{aligned}$$

□

2 Composition with Regular Functions

What happens when we take functions of a controlled rough path? Let $\varphi\in\mathcal{C}_b^2(W;\bar{W})$, $\boldsymbol{X}\in\mathcal{C}^\alpha(V)$, $(Y,Y')\in\mathcal{D}_X^{2\alpha}(\mathcal{L}(V;W))$. What is the Gubinelli derivative of $\varphi(Y)_t:=\varphi(Y_t)$? Natural guess is

$$\varphi(Y)'=\mathrm{D}\varphi(Y)Y'$$

which is consistent with composition of functions, in the sense that

$$\phi(\varphi(Y))'=\mathrm{D}\phi(\varphi(Y))\varphi(Y)'=\mathrm{D}(\phi\circ\varphi)(Y)Y'$$

Lemma 2. $(\varphi(Y),\varphi(Y)')\in\mathcal{D}_X^{2\alpha}(\mathcal{L}(V;\bar{W}))$.

Proof.
$$|\varphi(Y)_{s,t}|\leqslant\|\mathrm{D}\varphi\|_\infty|Y_{s,t}|\sim O(|t-s|^\alpha)$$

$$\begin{aligned} |\varphi(Y)_{s,t}'| &= |\mathrm{D}\varphi(Y_t)Y_t'-\mathrm{D}\varphi(Y_s)Y_s'|\\ &\leqslant \|\mathrm{D}\varphi\|_\infty|Y_{s,t}'|+\|Y'\|_\infty|\mathrm{D}\varphi(Y)_{s,t}|\\ &\leqslant \|\mathrm{D}\varphi\|_\infty|Y_{s,t}'|+\|Y'\|_\infty\|\mathrm{D}^2\varphi\|_\infty|Y_{s,t}|\sim O(|t-s|^\alpha) \end{aligned}$$

and

$$\begin{aligned} |R_{s,t}^\varphi| &= |\varphi(Y)_{s,t}-\varphi(Y)_s'X_{s,t}|\\ &= |\varphi(Y_t)-\varphi(Y_s)-\mathrm{D}\varphi(Y_s)Y_s'X_{s,t}|\\ &= |\varphi(Y_t)-\varphi(Y_s)-\mathrm{D}\varphi(Y_s)Y_{s,t}+\mathrm{D}\varphi(Y_s)R_{s,t}^Y|\\ &\lesssim \|\mathrm{D}^2\varphi\||Y_{s,t}|^2+\|\mathrm{D}\varphi(Y_s)\||R_{s,t}^Y|\sim O(|t-s|^{2\alpha}) \end{aligned}$$

□

Corollary 3. *(Leibniz).* $(Y, Y') \in \mathcal{D}_X^{2\alpha}, (Z, Z') \in \mathcal{D}_X^{2\alpha}$. *Then* $(YZ, Y'Z + YZ') \in \mathcal{D}_X^{2\alpha}$.

Proof.

$$|(Y'Z + YZ')_{s,t}| = |Y_t'Z_t + Y_tZ_t' - Y_s'Z_s - Y_sZ_s'| \sim O(|t-s|^{\alpha})$$

$$\begin{aligned} |R_{s,t}^U| &= |Y_tZ_t - Y_sZ_s - Y_s'Z_sX_{s,t} - Y_sZ_s'X_{s,t}| \\ &= |Y_tZ_t - Y_sZ_s - Y_{s,t}Z_s - Y_sZ_{s,t} + R_{s,t}^YZ_s + Y_sR_{s,t}^Z| \\ &\lesssim |Y_{s,t}Z_{s,t} + R_{s,t}^YZ_s + Y_sR_{s,t}^Z| \sim O(|t-s|^{2\alpha}) \end{aligned}$$

□

Lemma 4. $\varphi \in C_b^2(W; \bar{W})$, $X, Y \in \mathcal{C}^{\alpha}([0,T]; W)$. *Then, if* $\|X\|_{\alpha,[0,T]}, \|Y\|_{\alpha,[0,T]} \leqslant K$

$$\|\varphi(X) - \varphi(Y)\|_{\alpha;[0,T]} \lesssim_{\alpha,T,K} \|\varphi\|_{C_b^2}(|X_0 - Y_0| + \|X - Y\|_{\alpha;[0,T]})$$

Proof. Interpolate

$$\begin{aligned} \varphi(X_t) - \varphi(Y_t) = &= \int_0^1 \frac{\mathrm{d}}{\mathrm{d}r}\varphi(rX_t + (1-r)Y_t)\mathrm{d}r \\ &= \int_0^1 \mathrm{D}\varphi(rX_t + (1-r)Y_t)(X_t - Y_t)\mathrm{d}r \\ &= F(X_t, Y_t)(X_t - Y_t) \end{aligned}$$

so that

$$\begin{aligned} &|\varphi(X_t) - \varphi(Y_t) - \varphi(X_s) + \varphi(Y_s)| \\ &= |F(X_t, Y_t)(X_t - Y_t) - F(X_s, Y_s)(X_s - Y_s)| \\ &\leqslant |F(X_t, Y_t)(X_{s,t} - Y_{s,t})| + |(F(X_t, Y_t) - F(X_s, Y_s))(X_s - Y_s)| \\ &\lesssim \|\mathrm{D}\varphi\|_{\infty}|t-s|^{\alpha}\|X - Y\|_{\alpha} + \|\mathrm{D}^2\varphi\|_{\infty}(|X_{s,t}| + |Y_{s,t}|)\|X - Y\|_{\infty} \\ &\lesssim \|\varphi\|_{C_b^2}|t-s|^{\alpha}(\|X - Y\|_{\alpha} + K\|X - Y\|_{\infty}) \end{aligned}$$

so that claim follows because $\|X - Y\|_{\infty} \leqslant |X_0 - Y_0| + \sup_t(|X_{0,t} - Y_{0,t}|) \leqslant |X_0 - Y_0| + T^{\alpha}\|X - Y\|_{\alpha}$.

□

Recall that if $X, \tilde{X} \in \mathcal{C}^{\alpha}(V)$, $(Y, Y') \in \mathcal{D}_X^{2\alpha}(W)$, $(Y, Y') \in \mathcal{D}_{\tilde{X}}^{2\alpha}(W)$, we set a distance (not a metric, does not separate)

$$\|Y, Y'; \tilde{Y}, \tilde{Y}'\|_{X,\tilde{X},2\alpha} := \|Y' - \tilde{Y}'\|_{\alpha} + \|R^Y - R^{\tilde{Y}}\|_{2\alpha}$$

Then, same holds for controlled rough paths:

Theorem 5. *(Stability).*

$$\begin{aligned} &\|\varphi(Y) - \varphi(\tilde{Y})\|_{\alpha}, \|\varphi(Y), \varphi(Y)'; \varphi(\tilde{Y}), \varphi(\tilde{Y})'\|_{X,\tilde{X},2\alpha} \\ \lesssim &\|X - \tilde{X}\|_{\alpha} + |Y_0 - \tilde{Y}_0| + |Y_0' - \tilde{Y}_0'| + \|Y, Y'; \tilde{Y}, \tilde{Y}'\|_{X,\tilde{X},2\alpha} \end{aligned}$$

3 Ito's Lemma

We saw that an Ito's formula for $F\in C_b^3(V;W)$, $\boldsymbol{X}\in\mathcal{C}^\alpha(V)$:

$$F(X)_{0,t}=\int_0^t \mathrm{D}F(X_s)\mathrm{d}\boldsymbol{X}_s+\frac{1}{2}\int_0^t \mathrm{D}^2F(X_s)\mathrm{d}[\boldsymbol{X}]_s,$$

where $\mathcal{C}^{2\alpha}(V\otimes V)\ni[\boldsymbol{X}]_{s,t}:=X_{s,t}\otimes X_{s,t}-2\mathbb{S}_{s,t}$ (recall that $\delta[\boldsymbol{X}]_{s,u,t}=0$). We have a similar formula:

Theorem 6. *(Ito). $F\in C^3(V;W)$, $\boldsymbol{X}\in\mathcal{C}^\alpha$, $(Y,Y')\in\mathcal{D}_X^{2\alpha}$ of the form*

$$Y_t=Y_0+\int_0^t Y_s'\mathrm{d}\boldsymbol{X}_s+\Gamma_t$$

with $(Y',Y'')\in\mathcal{D}_X^{2\alpha}$, then

$$F(Y)_{0,t}=\int_0^t \mathrm{D}F(Y_s)Y_s'\mathrm{d}\boldsymbol{X}_s+\int_0^t \mathrm{D}F(Y_s)\mathrm{d}\Gamma_s+\frac{1}{2}\int_0^t \mathrm{D}^2(Y_s)(Y_s',Y_s')\mathrm{d}[\boldsymbol{X}]_s.$$

Proof. By the local approximation of the rough integral we have the increments

$$Y_{s,t}=Y_s'X_{s,t}+Y_s''\mathbb{X}_{s,t}+\Gamma_{s,t}+O(|t-s|^{3\alpha}) \tag{1}$$

We use Ito formula for rough paths $(Y,\mathbb{Y})\in\mathcal{C}^\alpha$

$$F(Y)_{u,v}=\mathrm{D}F(Y_u)Y_{u.v}+\mathrm{D}F(Y_u)'\mathbb{Y}_{u,v}+\mathrm{D}^2F(Y_u)[\boldsymbol{Y}]_{u,v}$$

Recall now that $\mathrm{D}F(Y_u)'=\mathrm{D}^2F(Y_u)$ and that $\mathbb{Y}_{u,v}=Y_u'Y_u'\mathbb{X}_{u,v}+O(|t-s|^{3\alpha})$ and that

$$\begin{aligned}[\boldsymbol{Y}]_{u,v} &= Y_{u,v}\otimes Y_{u,v}-2\mathrm{Sym}(\mathbb{Y}_{u,v})\\ &= Y_u'\mathbb{X}_{u,v}\otimes Y_u'\mathbb{X}_{u,v}-2Y_u'Y_u'\mathrm{Sym}(\mathbb{X}_{u,v})+O(|t-s|^{3\alpha})\\ &= Y_u'Y_u'[\boldsymbol{X}]_{u,v}+O(|t-s|^{3\alpha})\end{aligned}$$

we can write

$$\begin{aligned}F(Y)_{u,v} &= \mathrm{D}F(Y_u)Y_{u.v}+\mathrm{D}^2F(Y_u)Y_u'Y_u'\mathbb{X}_{u,v}+\mathrm{D}^2F(Y_u)Y_u'Y_u'[\boldsymbol{X}]_{u,v}+O(|t-s|^{3\alpha})\\ &= \mathrm{D}F(Y_u)(Y_{u.v}-Y_u''\mathbb{X}_{u,v})+\mathrm{D}F(Y_u)Y_u''\mathbb{X}_{u,v}\\ &\quad +\mathrm{D}^2F(Y_u)Y_u'Y_u'\mathbb{X}_{u,v}+\mathrm{D}^2F(Y_u)Y_u'Y_u'[\boldsymbol{X}]_{u,v}+O(|t-s|^{3\alpha})\\ &= \mathrm{D}F(Y_u)Y_u'X_{u,v}+(\mathrm{D}F(Y_u)Y_u''+\mathrm{D}^2F(Y_u)Y_u'Y_u')\mathbb{X}_{u,v}\\ &\quad +\mathrm{D}F(Y_u)\Gamma_{u,v}+\mathrm{D}^2F(Y_u)Y_u'Y_u'[\boldsymbol{X}]_{u,v}+O(|t-s|^{3\alpha})\end{aligned}$$

where we used the equation for Y, and the tools developed previously, e.g., the Leibniz rule and so on. The increment in the last line gives rise to Young integral since $\Gamma,[\boldsymbol{X}]\in\mathcal{C}^{2\alpha}$. □