Cluster Expansion Handout

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Definition 5.1

Sei Γ endl. Menge, $\gamma \in \Gamma$ heißen Polymere, mit

1. zu allen $\gamma \in \Gamma$ assoziiert man Gewicht (Aktivität) $\omega(\gamma) \in \mathbb{R}$ oder $\mathbb{C}$
2. Interaktionen zwischen Polymeren ist pw. codiert als symm. Funktion ($\delta(\gamma, \gamma') = \delta(\gamma', \gamma)$)
   δ: Γ x $\Gamma \to \mathbb{R}$, s.d.
   $\delta(\gamma, \gamma) = 0 \quad \forall\ \gamma \in \Gamma$
   $|\delta(\gamma, \gamma')| \leq 1 \quad \forall\ \gamma, \gamma' \in \Gamma$

Definition 5.2

$\Xi \coloneqq \sum_{\Gamma' \subset \Gamma} [\prod_{\gamma \in \Gamma} \omega(\gamma)] \cdot [\prod_{\{\gamma, \gamma'\} \subset \Gamma'} \delta(\gamma, \gamma')]$, wobei Γ' endl. Teilmenge von Γ, Ξ heißt (Polymer) Partitionsfunktion

Lemma 5.3

$\Xi = exp(\sum_{m \geq 1} \sum_{\gamma_1} \cdots \sum_{\gamma_m} \varphi(\gamma_1, ..., \gamma_m) \cdot \prod_{i \in V_m} (\gamma_i))$
wobei φ die Ursellfunktion ist.

Theorem 5.4

Ang. $|\delta(\gamma, \gamma')| \leq 1\ \ \forall \gamma, \gamma' \in \Gamma$, und $\exists\ a : \Gamma \to \mathbb{R}_+$, s.d. $\forall \gamma_* \in \Gamma : \sum_\gamma |\omega(\gamma)| e^{a(\gamma)} \cdot |\zeta(\gamma, \gamma_*)| \leq a(\gamma_*)$
Dann
$\forall \gamma_1 \in \Gamma : 1 + \sum_{k \geq 2} k \sum_{\gamma_2} \cdots \sum_{\gamma_k} |\varphi(\gamma_1, ..., \gamma_k| \cdot \prod_{j=2}^{k} |\omega(\gamma_i)| \leq e^{a(\gamma_1)}$
Insb. gilt (5.9)

Theorem 5.16

Sei $d \geq 2$. Dann:
$\exists 0 < \beta_0 < \infty, c > 0, C < \infty$, s.d.$\forall\ \beta \geq \beta_0$ gilt: $0 \leq\ < \sigma_i, \sigma_j >_{\beta,0}^{+}\ \leq C e^{-c\beta \cdot \|i-j\|_1} \quad \forall i, j \in \mathbb{Z}^d$

Alle obigen Ergebnisse basieren auf:
Sascha Friedli Sacha and Yvan Velenik, „Statistical Mechanics of Lattice Systems: A Concrete Mathematical Introduction“, Cambridge: Cambridge University Press, 2017

Referenzen

$Q[G] \coloneqq \sum_{\gamma_1} \cdots \sum_{\gamma_n} \prod_{i \in V} \omega(\gamma_i)\} \cdot \{\prod_{\{i,j\} \in E} \zeta(\gamma_i, \gamma_j)\}$
$\varphi(\gamma_1, ..., \gamma_n) \coloneqq \frac{1}{m!} \sum_{G \subset G_m zsh.} \prod_{\{i,j\} \in G} \zeta(\gamma_i, \gamma_j)$

(5.4) $\Xi = 1 + \sum_{n \geq 1} \frac{1}{n!} \sum_{\gamma_1} \cdots \sum_{\gamma_n} \{\prod_{i \in V_n} w(\gamma_i)\}\{\prod_{\{i,j\} \in E_n} \delta(\gamma_i, \gamma_j)\}$
(5.6) $\Xi = exp(\sum_{m \geq 1} \sum_{\gamma_1} \cdots \sum_{\gamma_m} \varphi(\gamma_1, ..., \gamma_m) \cdot \prod_{i \in V_m}(\gamma_i))$
(5.9) $\sum_{k \geq 1} \sum_{\gamma_1} \cdots \sum_{\gamma_k} \{|\varphi(\gamma_1, ..., \gamma_k)| \cdot \prod_{i=1}^{k} |w(\gamma_i)|\}$
(5.10) $\forall \gamma_* \in \Gamma : \sum_{\gamma} |\omega(\gamma)| e^{a(\gamma)} \cdot |\zeta(\gamma, \gamma_*)| \leq a(\gamma_*)$
(5.11) $\forall \gamma_1 \in \Gamma : 1 + \sum_{k \geq 2} k \sum_{\gamma_2} \cdots \sum_{\gamma_k} |\varphi(\gamma_1, ..., \gamma_k| \cdot \prod_{j=2}^{k} |\omega(\gamma_i)| \leq e^{a(\gamma_1)}$
(5.29) $\sum_{X : \bar{X} \ni i} |\psi(X)| \leq \sum_{S_1 \ni i} |w_h(S_1)| \cdot e^{|[S_1]_1|} \leq \eta(\Re(h), d) \leq 1$
(5.46) $\langle \sigma_A \rangle^{+}_{\beta,0} = exp\{\sum_{X \sim A} (\psi_{\beta}^{\{A\}}(X) - \psi_{\beta}(X))\}$
(5.47) $\langle \sigma_0 \rangle^{+}_{\beta,0} = exp\{\sum_{X \sim \{0\}} (\psi_{\beta}^{\{0\}}(X) - \psi_{\beta}(X))\}$
(5.48) $\langle \sigma_i, \sigma_j \rangle^{+}_{\beta,0} \geq \langle \sigma_i \rangle^{+}_{\beta,0} \langle \sigma_j \rangle^{+}_{\beta,0} = (\langle \sigma_0 \rangle^{+}_{\beta,0})^2 > 0$

Erinnerung

$\partial_e A \coloneqq \{\{i,j\} : i \sim j, \;\; i \in A, \;\; j \notin A\}$
$\Omega_{\Lambda}^{\eta} \coloneqq \{\omega \in \Omega : \omega_i = \eta_i, \forall i \notin \Omega\}$
$\mathscr{E}_{\Lambda}^{b} \coloneqq \{\{i,j\} \subset \mathbb{Z}^d : \{i,j\} \cap \Lambda \neq \emptyset, \;\; i \sim j\}$