Lectures on Stochastic Quantization of Φ^4_3

Part II: Stochastic quantization

M. Gubinelli (IAM & HCM, University of Bonn)

Online – February 15–25, 2021

This an extended version of the material presented in the second part of the course (10h). This script has not been seriously revised and contains typos and inconsistencies: use at your own risk. Also I didn't tried to cite all the literature associated to the material presented. Sometimes I cite sources where the reader can find more detailed discussions. These notes have been written with $\mathrm{T_EX_{MACS}}$.

1 Introduction

At the end of the first part of the course we introduced two measures $\mu^{L,\varepsilon}, \nu^{L,\varepsilon}$ defined as follows.

Let $\varepsilon = 2^{-N}$ and $M = 2^{N'}$. Let $\Lambda_\varepsilon = (\varepsilon\mathbb{Z})^d \subseteq \mathbb{R}^d$ the square lattice in dimension d of side length ε, $\Lambda_{\varepsilon,M} = \Lambda_\varepsilon \cap \mathbb{T}^d_M = (\varepsilon\mathbb{Z})^d \cap [-M/2, M/2)^d$ a finite box of $(M/\varepsilon + 1)^d$ points which we think with periodic boundary conditions in every directions.

Fourier transform on Λ_ε is defined as

$$\mathscr{F}_\varepsilon f(x) = \varepsilon^d \sum_{x \in \Lambda_\varepsilon} f(x) e^{-2\pi i k \cdot x}, \qquad \mathscr{F}_\varepsilon^{-1} g(x) = \int_{\hat{\Lambda}_\varepsilon} g(k) e^{2\pi i k \cdot x} \mathrm{d}k,$$

with $\hat{\Lambda}_\varepsilon = (\varepsilon^{-1}[-1,1))^d$ the dual of Λ_ε. These definitions can be extended to the finite lattice in a natural way, with $\hat{\Lambda}_{\varepsilon,M} = ((\mathbb{Z}/M) \cap [-\varepsilon^{-1}/2, \varepsilon^{-1}/2))^d$ and

$$\mathscr{F}_{\varepsilon,M} f(x) = \varepsilon^d \sum_{x \in \Lambda_{\varepsilon,M}} f(x) e^{-2\pi i k \cdot x}, \qquad \mathscr{F}_{\varepsilon,M}^{-1} g(x) = \frac{1}{M^d} \sum_{k \in \hat{\Lambda}_{\varepsilon,M}} g(k) e^{2\pi i k \cdot x}.$$

The measure μ is the law of a family of Gaussian r.v. $(X^{\varepsilon,M}(x))_{x \in \Lambda_{\varepsilon,M}}$ with covariance

$$\mathbb{E}_\mu[X^{\varepsilon,M}(x) X^{\varepsilon,M}(y)] = (m^2 - \Delta_\varepsilon)^{-1}(x,y), \qquad x, y \in \Lambda_{\varepsilon,M}$$

where Δ_ε is the discrete Laplacian with periodic boundary conditions, i.e.

$$\Delta_\varepsilon f(x) = \varepsilon^{-2} \sum_{i=1,\ldots,d} (f(x + \varepsilon e_i) - 2f(x) + f(x - \varepsilon e_i)), \qquad x \in \Lambda_\varepsilon$$

where $(e_i)_{i=1,\ldots,d}$ is the canonical basis of $\mathbb{R}^d$. We introduce also discrete derivatives

$$\nabla^i_\varepsilon f(x) = \frac{f(x + \varepsilon e_i) - f(x)}{\varepsilon}, \qquad \nabla^{-,i}_\varepsilon f(x) = \frac{f(x) - f(x - \varepsilon e_i)}{\varepsilon}$$

and note that $(\nabla^i_\varepsilon)^* = -\nabla^{-,i}_\varepsilon$ and $\Delta_\varepsilon = \sum_{i=1}^d \nabla^{-,i}_\varepsilon \nabla^i_\varepsilon$. Moreover

$$(\nabla^i_\varepsilon \mathscr{F}^{-1} g)(x) = \int_{\hat{\Lambda}_\varepsilon} g(k) \frac{e^{2\pi i \varepsilon k_i} - 1}{\varepsilon} e^{2\pi i k \cdot x} \mathrm{d}k$$

$$(\nabla_\varepsilon^{-,i}\nabla_\varepsilon^i\mathscr{F}^{-1} g)(x) = \int_{\hat{\Lambda}_\varepsilon} g(k)\frac{e^{2\pi i\varepsilon k_i}-1}{\varepsilon}\frac{1-e^{-2\pi i\varepsilon k_i}}{\varepsilon} e^{2\pi i k\cdot x}\mathrm{d}k = -\int_{\hat{\Lambda}_\varepsilon} g(k)(2\varepsilon^{-1}\sin(\pi\varepsilon k_i))^2 e^{2\pi i k\cdot x}\mathrm{d}k$$

A Fourier transform formula for the correlation function reads

$$(m^2-\Delta_\varepsilon)^{-1}(x,y) = \frac{1}{M^d}\sum_{k\in((\mathbb{Z}/M)\cap[-\varepsilon^{-1},\varepsilon^{-1}))^d}\frac{e^{ik\cdot(x-y)}}{(m^2+\sum_i(2\varepsilon^{-1}\sin(\pi\varepsilon k_i))^2)}, \qquad x,y\in\Lambda_{\varepsilon,M}$$

So $\varphi^{\varepsilon,M}$ is an approximation of the GFF φ. We denote $\mu^{\varepsilon,M}$ its law, note that it is a law on $\mathbb{R}^{\Lambda_{\varepsilon,M}}$ which is a finite dimensional space. By abuse of notation I will also consider it as a measure on $\mathbb{R}^{\Lambda_\varepsilon}$ by periodic extension.

Both our discrete versions of translation invariance and RP will converge nicely to their continuum counterpart. Finally

► Define the measure $\nu^{\varepsilon,M}$ on $\mathbb{R}^{\Lambda_{\varepsilon,M}}$ (or by extension on $\mathbb{R}^{\Lambda_\varepsilon}$)

$$\nu^{\varepsilon,M}(\varphi) = \frac{1}{Z_{\varepsilon,M}}\exp\left(-\varepsilon^d\sum_{x\in\Lambda_{\varepsilon,M}}V(\varphi(x))\right)\mu^{\varepsilon,M}(\mathrm{d}\varphi) \tag{1}$$

for some $V:\mathbb{R}\to\mathbb{R}$ bounded below.

Exercise. Prove that if $V(\varphi)=\beta\varphi^2$ and $\beta>-m^2$ then we get another GFF with a different mass.

This approximation now is elementary and it has the advantage that it preserves discrete translation invariance wrt. the lattice Λ_ε and moreover a discrete and periodic version of RP.

Reference for discrete RP: S. Friedli and Y. Velenik, *Statistical Mechanics of Lattice Systems: A Concrete Mathematical Introduction* (Cambridge, United Kingdom; New York, NY: Cambridge University Press, 2017).

►Both our discrete versions of translation invariance and RP will converge nicely to their continuum counterpart as $\varepsilon\to 0$ (to get rid of discreteness) and $M\to\infty$ (to get rid of periodicity).

The rest of the lectures will concern the analysis of these measures in order to prove the existence of the limits above.

What is a proper choice of V? Any V (non-quadratic) is ok, as soon as it works. The problem is that not so many choices are available. In $d=1$ one could take any $V\in C(\mathbb{R},\mathbb{R}_+)$ (or even unbounded with some conditions). In $d=2$ one can take polynomial functions, exponential, trigonometric functions. In $d=3$ we know only how to take V a fourth order polynomial bounded below, in this case we say we are looking at Φ_3^4.

Definition 1. *A Φ_3^4 measure is any non-Gaussian, Euclidean invariant and RP accumulation point of the family $(\nu^{\varepsilon,M})_{\varepsilon,M}$ as $\varepsilon\to 0$ and $M\to\infty$ where one can take as V any 4-th order polynomial, bounded below and with ε,M dependent coefficient.*

One of the big successes of constructive EQFT in '70,'80 is the proof that this limits exists and has many nice properties. It was proven by Glimm, Jaffe, Feldman, Osterwalder, Seneor, …

For ε fixed and $M \to \infty$ this is a problem of statistical mechanics: the infinite volume limit of a system of unbounded spins with nearest neighbor interaction.

As I said at the beginning a stochastic quantisation (in these lectures) of a given measure ρ is a map F_ρ which sends a Gaussian r.v. to a r.v. with law ρ.

Even in the case $\nu^{\varepsilon,M}$ there are many interesting ways to do this.

1. ***Langevin dynamics / parabolic SQ*** : the Gaussian process W is a family of Brownian motions and the map $\nu \sim F_\nu(W) = \phi(0)$ is given by the stationary solution

$$\phi: \mathbb{R} \times \Lambda_{\varepsilon,M} \to \mathbb{R},$$

of the SDE

$$\mathrm{d}\phi(t,x) = \{[(-m^2 + \Delta_\varepsilon)\,\phi(t)](x) - V'(\phi(t,x))\}\mathrm{d}t + \mathrm{d}W(t,x), \qquad x \in \Lambda_{\varepsilon,M}, t \in \mathbb{R}$$

with V' the derivative of V. Here $t \in \mathbb{R}$ is a fictious time (it is not the Euclidean time!!!)

2. ***Elliptic SQ***: $\nu \sim F_\nu(\xi) = (\phi(0,x))_{x\in\Lambda_{\varepsilon,M}}$ but now $\phi: \mathbb{R}^2 \times \Lambda_{\varepsilon,M} \to \mathbb{R}$ is the solution to the elliptic PDE

$$(m^2 - \Delta_{\mathbb{R}^2} - \Delta_{\Lambda_{\varepsilon,M}})\,\phi(z,x) + V'(\phi(z,x)) = \xi(z,x), \qquad x \in \Lambda_{\varepsilon,M}, z \in \mathbb{R}^2$$

where ξ is a space-time white noise.

3. ***Canonical SQ***: the Gaussian process W is a family of Brownian motions and the map $\nu \sim F_\nu(W) = \phi(0)$ is given by the stationary solution

$$\phi: \mathbb{R} \times \Lambda_{\varepsilon,M} \to \mathbb{R}$$

of the SDE (discrete wave equation)

$$\partial_t^2 \phi(t,x) = -\gamma\,\partial_t\phi(t,x) + [(-m^2 + \Delta_\varepsilon)\,\phi(t)](x) - V'(\phi(t,x)) + \partial_t W(t,x)$$

(approximatively). Without noise this is an Hamiltonian equation.

4. ***Variational representation*** (see Barashkov/G.)

5. There is even another possible approach which require to consider a stochastic evolution in the Euclidean time and it looks like

$$\partial_{x_0}\phi = \{-(m^2 - \Delta_\varepsilon)^{1/2}\,\phi - V'(\phi)\}\mathrm{d}t + \partial_{x_0}W, \qquad x \in \mathbb{R} \times \Lambda_{\varepsilon,M}^{d-1}.$$

In this case we cannot discretize the Euclidean time and also the measure $\nu^{\varepsilon,M}$ has to be taken slightly differently. This is essentially the Markovian point of view wrt. the EQFT we introduced in the first part of the course, where we perturb the OU process ϕ with a drift $-V'(\phi)$.

Remark. While the measure $\nu^{\varepsilon,M}$ is defined via a density wrt. to a Gaussian the goal of SQ is to define it as the push-forward of a Gaussian measure. In infinite dimensions it seems that push-forwards are more robust.

Example. Let $(B_t)_{t \geqslant 0}$ a one dim BM and let $X_t = B_t + t$. Then while there is no problem to see the law of X as push-forward of that of B, they are not absolutely continuous as measures on $C(\mathbb{R}_{\geqslant 0}; \mathbb{R})$.

Exercise. Prove it.

Historical note. Stochastic quantisation was introduced Nelson, Parisi & Wu. Rigorous construction of EQFT with stochastic quantisation was done in $d = 1$ by Jona–Lasinio and Faris ('80), Jona–Lasinio and Mitter (~'84) in $d = 2$ bounded volume and then Mitter et al. in infinite volume, this was done using probabilistic tools (martingale problems and Girsanov's formula). For $P(\Phi)_2$ (polynomial interaction in $d = 2$) another approach was introduced by Da Prato and Debussche. Only in 2013 Hairer managed to prove a local existence and uniqueness result for the parabolic SQ of Φ_3^4 using regularity structures. And the we had many more results… still many problems remain open.

References

For more details on the history of EQFT and SQ and a broad list of reference s look at the introductions of these papers:

- M. Gubinelli and M. Hofmanova, `A PDE Construction of the Euclidean Φ_3^4 Quantum Field Theory', *ArXiv:1810.01700 [Math-Ph]*, 3 October 2018, http://arxiv.org/abs/1810.01700.
- S. Albeverio, F. C. De Vecchi, and Massimiliano Gubinelli, `Elliptic Stochastic Quantization', *Annals of Probability* 48, no. 4 (July 2020): 1693–1741, https://doi.org/10.1214/19-AOP1404.
- S. Albeverio et al., `Grassmannian Stochastic Analysis and the Stochastic Quantization of Euclidean Fermions', *ArXiv:2004.09637 [Math-Ph]*, 25 May 2020, http://arxiv.org/abs/2004.09637.

Hyperbolic SQ and the variational method are discussed here:

- M. Gubinelli, H. Koch, and T. Oh, `Renormalization of the Two-Dimensional Stochastic Nonlinear Wave Equations', *Transactions of the American Mathematical Society*, 2018, 1, https://doi.org/10.1090/tran/7452.
- N. Barashkov and M. Gubinelli, `A Variational Method for Φ_3^4 ', *Duke Mathematical Journal* 169, no. 17 (November 2020): 3339–3415, https://doi.org/10.1215/00127094-2020-0029.

A broad literature on stochastic quantisation from the physicist point of view:

- Poul Henrik Damgaard and Helmuth Hüffel, *Stochastic Quantization* (World Scientific, 1988).

For an alternative approach to the infinite volume and infinite time limit of the dynamics:

- J. Dimock, `A Cluster Expansion for Stochastic Lattice Fields', *Journal of Statistical Physics* 58, no. 5 (1 March 1990): 1181–1207, https://doi.org/10.1007/BF01026571.

For a broad discussion of early applications of stochastic calculus in stochastic quantisation:

- Lorenzo Bertini, Giovanni Jona-Lasinio, and Claudio Parrinello, `Stochastic Quantization, Stochastic Calculus and Path Integrals: Selected Topics', *Progress of Theoretical Physics Supplement* 111 (1 January 1993): 83–113, https://doi.org/10.1143/PTPS.111.83.

Remark. An interesting recent talk of A. Jaffe "Is relativity compatible with quantum theory?"(December 2020) https://www.youtube.com/watch?v=RgQixyA2Gcs

It discusses the history and challenges in a mathematical theory of quantum fields.

2 Langevin dynamics

We start by constructing the parabolic stochastic quantization of the measure $\nu^{\varepsilon,M}$ for fixed ε,M. Since in this section these parameters do not play any role we will avoid to write the whenever it does not lead to ambiguities. In particular here Λ will denote the finite set $\Lambda_{\varepsilon,M}$ and Δ the discrete Laplacian and take $\varepsilon=1$ sometimes.

The law $\mu^{\varepsilon,M}$ is Gaussian, we can therefore introduce a fictious time $t\in\mathbb{R}$ (this is !not! the physical time) and a stationary OU process $(X_t^{\varepsilon,M})_{t\geqslant 0}$ such that $X_t^{\varepsilon,M}\sim\mu^{\varepsilon,M}$. There is not a unique choice, however it is not difficult to guess that a suitable dynamics is given by

$$\mathrm{d}X_t^{\varepsilon,M}=(\Delta_\varepsilon-m^2)X_t^{\varepsilon,M}\mathrm{d}t+2^{1/2}\mathrm{d}B_t^{\varepsilon,M}, \tag{2}$$

where $(B_t^{\varepsilon,M}(x))_{x\in\Lambda_{\varepsilon,M}}$ is a family of independent standard Brownian motions.

Exercise. Check the invariance of $\mu^{\varepsilon,M}$ under this dynamics, in particular pay attention to the normalization.

I want to construct now a dynamics which leave invariant the measure $\nu^{\varepsilon,M}$ instead. Let us guess what this dynamics should be: we write something similar as what we had before but with an unknown vector-field $F(t)$

$$\mathrm{d}X_t=AX_t\mathrm{d}t+F(t)\mathrm{d}t+2^{1/2}\mathrm{d}B_t.$$

with $A=(\Delta-m^2)$. Then if we denote $\mathbb{P}$ the law of the solution X of this equation with $X_0\sim\mu^{\varepsilon,M}$ and independent B, we want to have

$$\int_{\mathbb{R}^\Lambda}f(\varphi)\,\nu(\mathrm{d}\varphi)=\int_{\mathbb{R}^\Lambda}f(\varphi)\frac{e^{-U(\varphi)}}{Z}\,\mu(\mathrm{d}\varphi)=\frac{1}{Z}\mathbb{E}[f(X_0)\,e^{-U(X_0)}]=\frac{1}{Z}\mathbb{E}[f(X_t)\,e^{-U(X_0)}],$$

for all test functions f and all $t\geqslant 0$ with

$$U(\varphi)=\varepsilon^d\sum_{x\in\Lambda_{\varepsilon,M}}V(\varphi(x)),\qquad U\colon\mathbb{R}^{\Lambda_{\varepsilon,M}}\to\mathbb{R}.$$

Note that under the measure $\mathbb{P}^U$ defined as

$$\mathbb{P}^U(\mathrm{d}X)=\frac{e^{-U(X_0)}}{Z}\mathbb{P}(\mathrm{d}X),$$

the process X is still solution to the equation and $X_0\sim\nu^{\varepsilon,M}$.

► By Girsanov's formula, we have, for two test functions f, g

$$\mathbb{E}[f(X_t)\, e^{-U(X_0)}\, g(X_0)] = \mathbb{E}_{\mathbb{Q}}\Big[f(X_t)\, e^{\int_0^t F(s) 2^{1/2} \mathrm{d}W_s - \int_0^t |F(s)|^2 \mathrm{d}s - U(X_0)}\, g(X_0)\Big] \tag{3}$$

where under $\mathbb{Q}$ the process X satisfy the linear SDE

$$\mathrm{d}X_t = AX_t \mathrm{d}t + 2^{1/2} \mathrm{d}W_t$$

where W is a BM under $\mathbb{Q}$. Note that $X_0 \sim \mu$. So under $\mathbb{Q}$ X is an stationary OU process.

► Note that Ito formula gives

$$U(X_t) = U(X_0) + \int_0^t \mathrm{D}U(X_s) \mathrm{d}X_s + \int_0^t D^2 U(X_s) \mathrm{d}s$$

$$= U(X_0) + \int_0^t \mathrm{D}U(X_s)\, 2^{1/2} \mathrm{d}W_s + \int_0^t \underbrace{(\mathrm{D}^2 U(X_s) + \mathrm{DU}(X_s) AX_s)}_{=:Q(X_s)} \mathrm{d}s$$

so we can rewrite (3) as

$$= \mathbb{E}_{\mathbb{Q}}\Big[f(X_t)\, g(X_0)\, e^{-\frac{1}{2}U(X_t) - \frac{1}{2}U(X_0) + \frac{1}{2}\int_0^t (Q(X_s)\mathrm{d}s - |F(s)|^2)\mathrm{d}s}\, e^{\int_0^t F(s) 2^{1/2} \mathrm{d}W_s + \frac{1}{2}\int_0^t \mathrm{D}U(X_s) 2^{1/2} \mathrm{d}W_s}\Big]$$

and then take $F(s) = -\frac{1}{2}\mathrm{D}U(X_s)$ to cancel the stochastic integral in the exponent to get

$$\mathbb{E}[f(X_t)\, e^{-U(X_0)}\, g(X_0)] = \mathbb{E}_{\mathbb{Q}}\Big[f(X_t)\, g(X_0)\, e^{-\frac{1}{2}U(X_t) - \frac{1}{2}U(X_0) + \frac{1}{2}\int_0^t \left(Q(X_s)\mathrm{d}s - \frac{1}{4}|\mathrm{D}U(X_s)|^2\right)\mathrm{d}s}\Big]$$

and since $\mathbb{Q}$ is time reflection invariant (because under $\mathbb{Q}$ the process X is just a stationary OU process) we can rewrite this as

$$= \mathbb{E}_{\mathbb{Q}}\Big[f(X_0)\, g(X_t)\, e^{-\frac{1}{2}U(X_t) - \frac{1}{2}U(X_0) + \frac{1}{2}\int_0^t \left(Q(X_s)\mathrm{d}s - \frac{1}{4}|\mathrm{D}U(X_s)|^2\right)\mathrm{d}s}\Big]$$

where we exchanged the two functions. Taking $f = 1$ we have

$$\mathbb{E}_{\mathbb{Q}}\Big[g(X_t)\, e^{-\frac{1}{2}U(X_t) - \frac{1}{2}U(X_0) + \frac{1}{2}\int_0^t \left(Q(X_s)\mathrm{d}s - \frac{1}{4}|\mathrm{D}U(X_s)|^2\right)\mathrm{d}s}\Big] = \mathbb{E}[e^{-U(X_0)}\, g(X_0)]$$

and on the other hand, taking $g = 1$ we have (taking $g = f$ in the previous formula)

$$\mathbb{E}[f(X_t)\, e^{-U(X_0)}] = \mathbb{E}_{\mathbb{Q}}\Big[f(X_t)\, e^{-\frac{1}{2}U(X_t) - \frac{1}{2}U(X_0) + \frac{1}{2}\int_0^t \left(Q(X_s)\mathrm{d}s - \frac{1}{4}|\mathrm{D}U(X_s)|^2\right)\mathrm{d}s}\Big] = \mathbb{E}[e^{-U(X_0)}\, f(X_0)]$$

that is what we were looking for.

Remark. All this is ok provided we can perform all these computations. The only problems are related to the integrability of the exponential function involving the time integral. For example if we require that U is bounded below and moreover that

$$H(\varphi) = \frac{1}{2} Q(\varphi) - \frac{1}{8} |\nabla U(\varphi)|^2 = \mathrm{D}^2 U(\varphi) + \mathrm{DU}(\varphi) \cdot A\varphi - |\mathrm{D}U(\varphi)|^2$$

satisfies

$$\mathbb{E}_{\mathbb{Q}}\left[e^{\int_0^t H(X_s)\mathrm{d}s}\right] < \infty,$$

for some $t > 0$. Indeed it is enough to establish invariance for small time and the for all times.

►We learned that the solution to

$$\mathrm{d}X_t = AX_t\mathrm{d}t - \frac{1}{2}\mathrm{D}U(X_t)\mathrm{d}t + 2^{1/2}\mathrm{d}B_t \tag{4}$$

leaves the measure $\nu(\mathrm{d}\varphi) = Z^{-1} e^{-U(\varphi)} \mu(\mathrm{d}\varphi)$ invariant provided U is nice enough.

We will take this equation as stochastic quantization.

Exercise. Note that the process X is time-reversal invariant (we essentially gave a proof of this above, you can fill in the details).

We would actually like to have U which are unbounded, but bounded below, the relevant example in these lectures being

$$U(\varphi) = \varepsilon^d \sum_x \left(\frac{\lambda}{4}\varphi(x)^4 + \frac{\beta}{2}\varphi(x)^2\right)$$

for some $\lambda > 0$ and $\beta \in \mathbb{R}$.

We have two order of problems with such potentials. First

$$\mathrm{D}_x U(\varphi) = \frac{\partial}{\partial\varphi(x)} U(\varphi) = \lambda\,\varphi(x)^3 + \beta\,\varphi(x)$$

is not globally Lipschitz and the solutions to the SDE (4) could explode in finite time.

Then we still have to worry about invariance (i.e. fixing the details of the argument above) of the measure ν on this dynamics.

The second problem is merely technical and could be handled via a careful control of approximations with nice U and the above invariance argument. The first problem seems more worrisome but the key is to exploit the coercivity of the dynamics.

First method: one could use the invariance of the measure ν to conclude that solutions of the SDE do not explode, we will not do it here.

Second method: A direct approach is to test the equation with X_t, i.e. write

$$\frac{1}{2}\mathrm{d}\sum_x |X_t(x)|^2 = \sum_x X_t(x)\mathrm{d}X_t(x) + \sum_x \mathrm{d}t$$

$$= \sum_x [X_t(x)(AX_t)(x) - X_t(x)\mathrm{D}_x U(X_t)]\mathrm{d}t + 2^{1/2}\sum_x X_t(x)\mathrm{d}B_t(x) + \sum_x \mathrm{d}t$$

$$= -G(X_t)\mathrm{d}t + \beta\sum_x X_t(x)^2\mathrm{d}t + 2^{1/2}\sum_x X_t(x)\mathrm{d}B_t(x) + \sum_x \mathrm{d}t$$

with in the polynomial case (summing by parts the Laplacian)

$$G(\varphi)=\sum_x (|\nabla_\varepsilon\varphi(x)|^2+m^2\varphi(x)^2+\lambda\varphi(x)^4)\geqslant 0.$$

By taking averages we could get some interesting estimates, for example

$$\mathbb{E}\sum_x |X_t(x)|^2+\int_0^t G(X_s)\mathrm{d}s=\mathbb{E}\,\beta\int_0^t\sum_x X_s(x)^2\mathrm{d}s+\sum_x \mathrm{d}t,$$

where now the r.h.s. can be controlled via the l.h.s. or via Gronwall lemma. But this is not robust enough for what is going on next week.

Third and last method: a more elementary and useful in the following approach which do not rely on Ito's formula goes as follows (this essentially what is called the Da Prato–Debussche trick).

First one write $X=Y+Z$ where Y is the solution to the linear equation

$$\mathrm{d}Y_t=A\,Y_t\mathrm{d}t+2^{1/2}\mathrm{d}B_t,$$

that is an OU process, and Z is what remains. Then Z must solve

$$\frac{\mathrm{d}Z_t}{\mathrm{d}t}=\left(AZ_t-\frac{1}{2}\nabla U(Y_t+Z_t)\right) \tag{5}$$

which is an ODE with random coefficients, not a stochastic differential equation anymore since the effect of the Brownian perturbation is completely taken into account by Y.

We can now test this equation with Z (without the need of Ito's formula) and obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\sum_x |Z_t(x)|^2+G(Z_t)=\sum_x \lambda\,(Y_t(x)^3Z_t(x)+3\,Y_t(x)^2Z_t(x)^2+3\,Y_t(x)\,Z_t(x)^3)$$

$$+\beta\sum_x (Z_t(x)\,Y_t(x)+Z_t(x)^2)$$

where

$$G(\varphi)=\|\nabla\varphi\|_{L^2}^2+m^2\|\varphi\|_{L^2}^2+\lambda\|\varphi\|_{L^4}^4,$$

with the natural Lebesgue spaces on $\Lambda=\Lambda_{\varepsilon,M}$ (with counting measure).

The key property being that in the r.h.s. we have all terms which we can bound via Hölder inequality as

$$\frac{\mathrm{d}}{\mathrm{d}t}\sum_x |Z_t(x)|^2+G(Z_t)\leqslant C_\delta\|Y_t\|_{L^4}^4+\delta G(Z_t),$$

for $\delta>0$ small as we wish, e.g. $\delta=1/2$. We conclude that

$$\|Z_t\|_{L^2}^2+\frac{1}{2}\int_0^t G(Z_s)\mathrm{d}s\leqslant\|Z_0\|_{L^2}^2+C\int_0^t\|Y_s\|_{L^4}^4\mathrm{d}s. \tag{6}$$

This bound implies that solutions cannot explode and we have an explicit bound on its growth in term of Y and Z_0.

Of the two we know very well Y (it is the OU process, it is Gaussian, I know everything I want on it). On the other hand I do not know so well

$$Z_0 = X_0 - Y_0 \sim \text{“}\nu - \mu\text{”}$$

because we do not really know very well ν (which is actually the object we want to study). For example we do not know estimates uniform in ε, M.

Note that even if X is stationary and Y is stationary (because we take $X_0 \sim \nu$ and $Y_0 \sim \mu$ and independent). But they are not independent and more importantly Z is not stationary.

One would like to prove that there exists a coupling of X_0 and Y_0 (i.e. find a joint law with marginals ν and μ respectively) so that the process (X, Y) is stationary (as a pair) from which would follow that Z is stationary.

In any case what we have so far is that for any f and any t we have

$$\int f(\varphi)\, \nu(\mathrm{d}\phi) = \mathbb{E}[f(X_t)] = \frac{1}{t} \int_0^t \mathbb{E}[f(X_s)]\mathrm{d}s = = \frac{1}{t} \int_0^t \mathbb{E}[f(Y_s + Z_s)]\mathrm{d}s$$

using stationarity. This is the stochastic quantization equation. Estimates on X are given via Y and Z.

Let's construct a stationary coupling of Y and Z. One uses the Krylov-Bogoliubov argument. We can construct a measure γ_T on a pair of fields $(\varphi, \psi) \in \mathbb{R}^\Lambda \times \mathbb{R}^\Lambda$ by the formula

$$\int f(\varphi, \psi)\mathrm{d}\gamma_T(\varphi, \psi) := \frac{1}{T} \int_0^T \mathbb{E}[f(Y_s, Z_s)]\mathrm{d}s,$$

for any bounded function f of the pair $(\varphi, \psi) \in \mathbb{R}^\Lambda \times \mathbb{R}^\Lambda$ where Y, Z are started as above.

We have that

$$\int [G(\psi) + \|\varphi\|_{L^4}^4]\mathrm{d}\gamma_T(\varphi, \psi) = \frac{1}{T} \int_0^T \mathbb{E}[G(Z_s) + \|Y_s\|_{L^4}^4]\mathrm{d}s \leqslant \frac{2}{T}\left(\mathbb{E}\|Z_0\|_{L^2}^2 + C' \int_0^T \mathbb{E}\|Y_s\|_{L^4}^4 \mathrm{d}s\right),$$

$$\leqslant \left(\frac{2}{T}\mathbb{E}\|Z_0\|_{L^2}^2\right) + 2C'\, \mathbb{E}\|Y_0\|_{L^4}^4,$$

which is uniformly bounded in T. This implies that the family $(\gamma_T)_T$ is tight on $\mathbb{R}^\Lambda \times \mathbb{R}^\Lambda$ and one can extract a weakly convergent subsequence to a limit γ. Note also that

$$\int f(\varphi + \psi)\mathrm{d}\gamma_T(\varphi, \psi) = \frac{1}{T} \int_0^T \mathbb{E}[f(Y_s + Z_s)]\mathrm{d}s = \frac{1}{T} \int_0^T \mathbb{E}[f(X_s)]\mathrm{d}s = \mathbb{E}[f(X_0)].$$

Therefore the law of $\varphi + \psi$ under γ_T is always given by ν for any T. As a consequence the law of $\varphi + \psi$ under γ is ν.

The measure γ is stationary under the joint dynamics of (Z, Y), i.e. if $(Z_0, Y_0) \sim \gamma$ then $(Z_t, Y_t) \sim \gamma$.

Exercise. Prove it. Also try to understand if the dynamics of the pair is time-symmetric.

In this way one can construct a stationary coupling of (Z, Y) which gives a useful representation of the stationary process X.

3 Infinite volume limit

What happens when we want to take the limit $M\to\infty$? The estimate (6) is not good enough because both $\|Z_0\|_{L^2(\Lambda_{\varepsilon,M})}$ and $\|Y_s\|_{L^4(\Lambda_{\varepsilon,M})}$ cannot remain finite since both random field are stationary and one expects that

$$\|Z_0\|_{L^2(\Lambda_{\varepsilon,M})}\sim M^d, \qquad \|Y_s\|_{L^4(\Lambda_{\varepsilon,M})}\sim M^d.$$

In this section we explicit the dependence on M and use Λ_ε for the full lattice. Moreover we extend any periodic field to the full lattice periodically. (we fix an origin)

However we can modify our apriori estimate introducing a polynomial weight $\rho\colon\Lambda=(\varepsilon\mathbb{Z})^d\to\mathbb{R}$

$$\rho(x)=(1+\ell|x|)^{-\sigma}$$

with $\ell,\sigma>0$ large enough, where $|x|$ is the distance from the origin of $\Lambda=\Lambda_\varepsilon$.

Now we test the equation for Z with $\rho^2 Z$ summing over the full lattice Λ and we get

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\sum_{x\in\Lambda_\varepsilon}|\rho(x)Z_t(x)|^2+G(Z_t)\leqslant-\lambda\sum_{x\in\Lambda_\varepsilon}\rho(x)(Y_t(x)^3Z_t(x)+3Y_t(x)^2Z_t(x)^2+3Y_t(x)Z_t(x)^3)$$

$$+\beta\sum_{x\in\Lambda_\varepsilon}\rho(x)(Z_t(x)Y_t(x)+Z_t(x)^2)+C_\rho\sum_{x\in\Lambda_\varepsilon}\rho(x)Z_t(x)^2$$

where C_ρ (and the inequality) is term coming from the integration by parts which can be made small by choosing ℓ small and where

$$G(\varphi)=\|\rho\nabla\varphi\|_{L^2(\Lambda_\varepsilon)}^2+m^2\|\rho\varphi\|_{L^2(\Lambda_\varepsilon)}^2+\lambda\|\rho^{1/2}\varphi\|_{L^4(\Lambda_\varepsilon)}^4.$$

And using similar estimates as above we obtain the apriori weighted estimates:

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\rho Z_t\|_{L^2(\Lambda_\varepsilon)}^2+G(Z_t)\leqslant C_\delta\|\rho^{1/2}Y_t\|_{L^4(\Lambda_\varepsilon)}^4+\delta G(Z_t)$$

indeed

$$\lambda\left|\sum_{x\in\Lambda_\varepsilon}\rho(x)Y_t(x)^3Z_t(x)\right|\leqslant\lambda\left|\sum_{x\in\Lambda_\varepsilon}(\rho(x)^{3/2}Y_t(x)^3)(\rho(x)^{1/2}Z_t(x))\right|$$

$$\leqslant\lambda\frac{C}{\delta}\|\rho^{1/2}Y_t\|_{L^4}^4+\delta\lambda\|\rho^{1/2}Z_t\|_{L^4}^4\leqslant\lambda\frac{C}{\delta}\|\rho^{1/2}Y_t\|_{L^4}^4+\delta G(Z_t)$$

for any small $\delta>0$.

As a consequence one get the estimate

$$\|\rho Z_t\|_{L^2(\Lambda)}^2+\frac{1}{2}\int_0^t G(Z_s)\mathrm{d}s\leqslant\|\rho Z_0\|_{L^2(\Lambda)}^2+C\int_0^t\|\rho^{1/2}Y_s\|_{L^4(\Lambda)}^4\mathrm{d}s. \qquad (7)$$

We have seen that we can construct a stationary coupling of (Y,Z), so we can use there this stationary coupling and take the average of this inequality to get

$$\mathbb{E}\|\rho Z_t\|_{L^2(\Lambda)}^2+\frac{1}{2}\int_0^t\mathbb{E}\,G(Z_s)\mathrm{d}s\leqslant\mathbb{E}\|\rho Z_0\|_{L^2(\Lambda)}^2+C\int_0^t\mathbb{E}\|\rho^{1/2}Y_s\|_{L^4(\Lambda)}^4\mathrm{d}s$$

but by stationarity we also have $\mathbb{E}\|\rho Z_t\|_{L^2(\Lambda)}^2 = \mathbb{E}\|\rho Z_0\|_{L^2(\Lambda)}^2$ so the initial condition disappear!!! So

$$\frac{1}{2}\int_0^t \mathbb{E}\, G(Z_s)\mathrm{d}s \leqslant C\int_0^t \mathbb{E}\|\rho^{1/2} Y_s\|_{L^4(\Lambda)}^4 \mathrm{d}s$$

and again by stationarity one get

$$\mathbb{E}\, G(Z_0) \leqslant 2C\,\mathbb{E}\|\rho^{1/2} Y_0\|_{L^4(\Lambda)}^4.$$

Which give us very good apriori estimates on the law of Z_0 which are independent of M, indeed

$$\mathbb{E}\|\rho^{1/2} Y_0\|_{L^4(\Lambda)}^4 = \mathbb{E}\sum_{x\in\Lambda_\varepsilon} \rho(x)^2 |Y_0(x)|^4 = \sum_{x\in\Lambda_\varepsilon} \rho(x)^2\, \mathbb{E}|Y_0(x)|^4 = C\sum_{x\in\Lambda_\varepsilon} \rho(x)^2 < \infty$$

uniformly in M provided $\sigma > d$ and the law of $Y_0(x)$ is translation invariant so does not depend on x and actually one can easily show that

$$(\mathbb{E}|Y_0(x)|^4)^{1/2} \leqslant C\,\mathbb{E}|Y_0(x)|^2$$

$$\lesssim (m^2-\Delta)^{-1}(x,x) \lesssim \frac{1}{M^d} \sum_{k\in((\mathbb{Z}/M)\cap[-\varepsilon^{-1},\varepsilon^{-1}))^d} \frac{1}{(m^2+\sum_i (2\,\varepsilon^{-1}\sin(\pi\varepsilon k_i))^2)}$$

$$\to \int_{[-\varepsilon^{-1},\varepsilon^{-1})^d} \frac{1}{(m^2+\sum_i (2\,\varepsilon^{-1}\sin(\pi\varepsilon k_i))^2)} < +\infty$$

uniformly in M.

Lemma 2. *For any $M>0$ we have that $\nu^M \sim X_0^M \sim Y_0^M + Z_0^M$ where $Y_0^M \sim \mu^M$ and Z_0^M is a r.v. such that*

$$\sup_M \int_{\mathbb{R}^\Lambda} G(\varphi)\,\nu^M(\mathrm{d}\varphi) = \sup_M \mathbb{E}\, G(Z_0^M) < \infty.$$

This is a key estimate to take the infinite volume limit since it allows to use tightness on the family $(\nu^M)_M$ on $\mathbb{R}^\Lambda$ in the topology of local convergence.

Remark. Integration by parts with a weight was treated a bit sloppily above. Let us make these considerations more precise: we have

$$\varepsilon\nabla_\varepsilon^i(fg)(x) = f(x+\varepsilon e_i)\,g(x+\varepsilon e_i) - f(x)\,g(x) = (\nabla_\varepsilon^i f(x) + f(x))\,g(x+\varepsilon e_i) - f(x)\,g(x)$$

$$= \nabla_\varepsilon^i f(x)\,g(x+\varepsilon e_i) + f(x)\,\nabla_\varepsilon^i g(x)$$

so

$$\begin{aligned} -\sum_x \rho^2 Z\,\Delta Z &= \sum_x\sum_i \nabla^i(\rho^2 Z)\nabla^i Z = \sum_x\sum_i \rho^2|\nabla^i Z|^2 + \sum_x\sum_i \nabla^i(\rho^2)\,Z(\cdot+\varepsilon e_i)\nabla^i Z \\ &= \sum_x\sum_i \rho^2|\nabla^i Z|^2 + \sum_x\sum_i \nabla^i(\rho^2)\,Z(\cdot+\varepsilon e_i)\nabla^i Z \end{aligned} \qquad (8)$$

and the second term can be bounded by

$$\left|\sum_x \sum_i \nabla^i(\rho^2) Z(\cdot+\varepsilon e_i)\nabla^i Z\right| \leqslant C\|\nabla^i(\rho^2)\,\rho^{-2}\|_{L^\infty}\|\nabla Z\|_{L^2}\|Z\|_{L^2} \leqslant C_\rho\|\nabla Z\|_{L^2}^2 + C_\rho\|Z\|_{L^2}^2$$

with $C_\rho = C'\|\nabla^i(\rho^2)\,\rho^{-2}\|_{L^\infty}$. Note that this constant can be made as small as we want (uniformly in ε, M) by taking ℓ small, indeed this is essentially a discrete version of the fact that in the continuum ($\varepsilon \to 0$) one has

$$|\nabla(\rho^2)\,\rho^{-2}| = |\rho^{-1}\nabla\rho| = (1+\ell|x|)^{-1}\,\ell \lesssim \ell.$$

This argument shows however that one has to be careful with weights which are compactly supported since the term $\rho^{-1}\nabla\rho$ is then more tricky to estimate. For example, in one dimension if one has a function $\rho(t) \approx t^\alpha$ with goes to zero as $t \to 0$ then

$$\rho^{-1}\nabla\rho \propto t^{-\alpha}\,t^{\alpha-1} \propto t^{-1}$$

so the above L^∞ estimate is not true anymore. For our purpose here polynomial weights are enough.

More estimates can be obtained by testing with other functions. For example, testing (5) with $\rho(\rho Z)^{p-1}$ one gets

$$\frac{1}{p}\partial_t \sum_\Lambda (\rho Z_t)^p + \sum_\Lambda m^2(\rho Z_t)^p + \rho(\rho Z_t)^{p-1}\,\Delta Z_t + \lambda(\rho^{p/(p+2)} Z_t)^{p+2}$$
$$= -\lambda \sum_\Lambda \rho(\rho Z_t)^{p-1}[(Y_t + Z_t)^3 - Z_t^3]$$

and by proceeding as above one obtains uniform weighted $L^p(\Lambda)$ estimates for ν^M.

Remark 3. It is also possible to obtain weighted L^∞ estimates.

These weighted estimates are uniform in M and allow to prove tightness of the family $(\nu^{\varepsilon,M})_M$ for fixed $\varepsilon > 0$ and $M \to \infty$, in the topology of local convergence (i.e. convergence by testing with continuous functions on $\mathbb{R}^{\Lambda_\varepsilon}$ which depends only of finitely many points of Λ_ε). In particular we understood that the local (or weighted) $L^p(\Lambda_\varepsilon)$ norms of $\varphi: \mathbb{R}^{\Lambda_\varepsilon} \to \mathbb{R}$ under the measure $\nu^{\varepsilon,M}$ have finite moments:

$$\sup_M \int \|\rho\,\varphi\|_{L^p}^p\,\nu^{\varepsilon,M}(\mathrm{d}\varphi) < \infty$$

for any $p > 1$. Actually by working a bit harder one can prove uniform integrability of functions like $\exp(\|\rho\,\varphi\|_{L^2})$.

4 Convergence to equilibrium

At this point we want also to probe the behavior of the dynamics for long times and also large distances.

The appropriate way to look at this problem is to imagine that we have two solutions Z^1 and Z^2 both driven by two OU processes Y^1, Y^2 and with arbitrary initial conditions Z_0^1, Z_0^2.

We will fix later the specific ways we choose these data, according to the property of the measures we would like to establish.

For the moment the only thing we care about is that their difference $H = Z^1 - Z^2$ solves the equation

$$\partial_t H - AH = -\frac{1}{2}[U'(X^1) - U'(X^1 + H + K)] =: Q$$

with $K := Y^1 - Y^2$ and $X^1 = Y^1 + Z^1$ as usual.

The r.h.s. can be Taylor-expanded as

$$Q = -\frac{1}{2}\int_0^1 \mathrm{d}\tau\, U''(X^1 + \tau(H+K))(H+K).$$

Note that for our usual expression of U we have

$$U_x''(\varphi) = (4\cdot 3)\,\lambda\,\varphi^2 + 2\,\beta.$$

However we will not need here this specific expression. We can deal with a more general situation.

We need however to concentrate on estimates which are good when $H, K \ll 1$, since we want to show that the two solutions are near when the difference in the data, i.e. in K and H_0 is small.

We will make there the key hypothesis that

$$U''(\varphi) \geqslant -2\,\chi$$

for some constant $\chi \in \mathbb{R}$.

This a convexity assumption since it implies that $\varphi \mapsto U(\varphi) + 2\,\chi\,\sum_\Lambda \varphi^2$ is convex. (I'm a bit sloppy here about the precise meaning of the derivatives, but we are dealing with local functionals, so I leave to you to fill out the details)

We also let

$$G := \frac{1}{2}\int_0^1 \mathrm{d}\tau\, U''(X^1 + \tau(H+K)) \geqslant -\chi.$$

We test the equation with $\rho^2 H$ for some weight ρ. The r.h.s. can be bounded by (δ is small)

$$\sum_\Lambda \rho^2 HQ = \sum_\Lambda \rho^2 GKH - \sum_\Lambda \rho^2 GH^2$$

$$\leqslant C_\delta \|\rho GK\|_{L^2}^2 + \delta\|\rho H\|_{L^2}^2 - \underbrace{\sum_\Lambda \rho^2 GH^2}_{\geqslant \chi \|\rho H\|_{L^2}^2}$$

$$\leqslant C_\delta \|\rho GK\|_{L^2}^2 + (\chi + \delta)\|\rho H\|_{L^2}^2$$

So we have now, taking any $\delta \leqslant \chi$,

$$\frac{1}{2}\partial_t \|\rho H\|_{L^2}^2 + \sum_{\Lambda} \rho^2 H\,(m^2 - \Delta)\,H \leqslant C\,\|\rho\,G K\|_{L^2}^2 + 2\,\chi\,\|\rho H\|_{L^2}^2$$

Consider furthermore $F(t) = e^{ct}\|\rho H\|_{L^2}^2$, then

$$\frac{1}{2}\partial_t(e^{ct}\|\rho H\|_{L^2}^2) = \frac{c}{2}(e^{ct}\|\rho H\|_{L^2}^2) + \frac{e^{ct}}{2}\partial_t\|\rho H\|_{L^2}^2$$

$$\leqslant -e^{ct}\sum_{\Lambda} \rho^2 H\left(m^2 - 2\,\chi - \frac{c}{2} - \Delta\right)H + e^{ct}\,C\,\|\rho\,G K\|_{L^2}^2.$$

Let

$$\tilde{Q}(t) := C\,\|\rho\,G\,(t) K(t)\|_{L^2}^2.$$

Choose a different kind of weight here, exponential, in the form

$$\rho\,(x) = e^{-\theta|x|},$$

then for $\varepsilon\,\theta \leqslant 1$ we have

$$|\nabla^i(\rho^2)| = \left|\frac{e^{-\theta|x+\varepsilon e_i|} - e^{-\theta|x|}}{\varepsilon}\right| \leqslant \frac{e^{\varepsilon\theta} - 1}{\varepsilon}\,e^{-\theta|x|} \leqslant 2\,\theta\,e^{-\theta|x|}$$

and using (8) we get

$$\sum_{\Lambda} \rho^2 H\,(-\Delta)H \geqslant \sum_{\Lambda}\sum_{i} \rho^2|\nabla_i H|^2 - \sum_{\Lambda}\sum_{i} 4\,\theta\,\rho^2|H(\cdot + \varepsilon\,e_i)||\nabla^i H|$$

$$\geqslant \sum_{\Lambda}\sum_{i} \rho^2|\nabla_i H|^2 - C\,\theta^2\sum_{\Lambda} \rho^2|H|^2 - \frac{1}{2}\sum_{\Lambda}\sum_{i} \rho^2|\nabla_i H|^2$$

$$\geqslant \frac{1}{2}\sum_{\Lambda}\sum_{i} \rho^2|\nabla_i H|^2 - C\,\theta^2\sum_{\Lambda} \rho^2|H|^2.$$

Putting all together this gives

$$\frac{1}{2}\partial_t(e^{ct}\|\rho H\|_{L^2}^2) + e^{ct}\left(m^2 - 2\,\chi - \frac{c}{2} - C\,\theta^2\right)\sum_{\Lambda} \rho^2 H^2 + e^{ct}\frac{1}{2}\sum_{\Lambda}\sum_{i} \rho^2|\nabla_i H|^2$$

$$\leqslant \frac{1}{2}\partial_t(e^{ct}\|\rho H\|_{L^2}^2) + e^{ct}\sum_{\Lambda} \rho^2 H\left(m^2 - \frac{c}{2} - 2\,\chi\right)H + e^{ct}\sum_{\Lambda} \rho^2 H\,(-\Delta)H \leqslant e^{ct}\,\tilde{Q}(t).$$

Assuming that

$$m^2 - 2\,\chi - \frac{c}{2} - C\,\theta^2 \geqslant 0$$

we have

$$\frac{1}{2}\partial_t(e^{ct}\|\rho H\|_{L^2}^2) \leqslant e^{ct}\,\tilde{Q}(t).$$

Integrating this we conclude

$$\frac{e^{ct}}{2}\|\rho H_t\|_{L^2}^2 \leqslant \left[\frac{1}{2}\|\rho H_0\|_{L^2}^2 + \int_0^t e^{cs}\tilde{Q}(s)\mathrm{d}s\right]$$

that is

$$\|\rho H_t\|_{L^2}^2 \leqslant e^{-ct}\|\rho H_0\|_{L^2}^2 + 2\int_0^t e^{-c(t-s)}\tilde{Q}(s)\mathrm{d}s$$

where

$$\tilde{Q}(s) = C\|\rho G_s K_s\|_{L^2}^2 = C\int_0^1 \mathrm{d}\tau \|\rho^2 U''(X_s^1 + \tau(H_s + K_s))^2 K_s^2\|_{L^1(\Lambda)}$$

For example, taking averages of this source term we can estimate it as

$$\mathbb{E}[\tilde{Q}(s)] \leqslant \sum_{x\in\Lambda}\int_0^1 \mathrm{d}\tau\, \rho^2(x)\left\{\mathbb{E}\left[U_x''(X_s^1 + \tau(H_s + K_s))^4\right]\right\}^{1/2}\left(\mathbb{E}\, K_s^4(x)\right)^{1/2}$$

From our estimates above for Z^1, Z^2, Y^1, Y^2 is not difficult to deduce that

$$\mathbb{E}\left[U_x''(X_s^1 + \tau(H_s + K_s))^4\right] \leqslant C$$

uniformly in M for any polynomial U''. When K is stationary in time we have the simpler expression

$$\mathbb{E}[\tilde{Q}(s)] \lesssim \sum_{x\in\Lambda}\rho^2(x)(\mathbb{E}\, K_s^4(x))^{1/2} \lesssim \sum_{x\in\Lambda}\rho^2(x)(\mathbb{E}\, K_0^4(x))^{1/2}.$$

We summarize these computation as

Lemma. *Uniformly in M and provided $m^2 - 2\chi - \frac{c}{2} - C\theta^2 \geqslant 0$ and K is stationary in time we have*

$$\|\rho H_t\|_{L^2}^2 \leqslant e^{-ct}\|\rho H_0\|_{L^2}^2 + C\sum_{x\in\Lambda}\rho^2(x)(\mathbb{E}\, K_0^4(x))^{1/2}.$$

Remark. This estimate shows that as $t\to\infty$ and provided

$$\sum_{x\in\Lambda}\rho^2(x)(\mathbb{E}\, K_0^4(x))^{1/2} \to 0 \tag{9}$$

we have

$$\|\rho H_t\|_{L^2}^2 \to 0.$$

Lemma 4 can be used in two different ways: by coupling two different invariant measures via a common dynamics one can show that the two measures are equal. This gives uniqueness. On can use noises which coincide in a bounded region to drive two different dynamics, e.g. started from the same invariant measure. Then one obtains that the quantity (9) can be made small choosing a large region, which shows that one has a certain decay of correlations, i.e. what happens outside a given region does not influence much the dynamics.

5 The small scale limit and renormalization ($d = 2$)

In this section we address the $\varepsilon \to 0$ limit at M fixed (let's say $M = 1$). This is the ultraviolet limit (UV limit). Obtain uniform estimates in this limit is more difficult are requires new ideas. There are various possible approaches: regularity structures (Hairer), renormalization group ideas (Kupiainen), or paracontrolled distributions (GIP, Catellier & Chouk). I will follow this last strategy. The main reference for us here is the paper I mentioned by Hofmanova & myself:

> M. Gubinelli and M. Hofmanova, `A PDE Construction of the Euclidean Φ_3^4 Quantum Field Theory', *ArXiv:1810.01700 [Math-Ph]*, 3 October 2018, http://arxiv.org/abs/1810.01700.

The main problem is that as $\varepsilon \to 0$ the process Y becomes a distribution. Recall our context. We had a dynamics on X which can be decomposed on a linear part

$$\mathrm{d}Y_t = (\Delta_\varepsilon - m^2)\, Y_t \mathrm{d}t + 2^{1/2} \mathrm{d}B_t,$$

and the non-linear part Z:

$$\frac{\partial}{\partial t} Z_t = (\Delta_\varepsilon - m^2)\, Z_t - \frac{1}{2} V'(Y_t + Z_t), \tag{10}$$

with $V'(\varphi) = \lambda \varphi^3 + \beta \varphi$. The computation of $V'(Y_t + Z_t)$ is point-wise in space:

$$V'(Y_t + Z_t)(x) = V'(Y_t(x) + Z_t(x)) = \lambda (Y_t(x) + Z_t(x))^3 + \beta (Y_t(x) + Z_t(x))$$

$$= \lambda Y_t(x)^3 + 3 \lambda Y_t(x)^2 Z_t(x) + 3 \lambda Y_t(x) Z_t(x)^2 + \lambda Z_t(x)^3 + \beta Y_t(x) + \beta Z_t(x).$$

The main problem is the following: ($M = 1$) as $\varepsilon \to 0$

$$\mathbb{E}[Y_t(x)^2] = (m^2 - \Delta_\varepsilon)(x, x) = \sum_{k \in (\mathbb{Z} \cap [-\varepsilon^{-1}, \varepsilon^{-1}))^d} \frac{1}{(m^2 + \sum_i (2 \varepsilon^{-1} \sin(\pi \varepsilon k_i))^2)}$$

$$\approx \sum_{k \in \mathbb{Z} \cap [-\varepsilon^{-1}, \varepsilon^{-1})^d} \frac{1}{(m^2 + 2\pi |k|^2)} \propto \begin{cases} \varepsilon^{2-d} & d > 2 \\ \log(\varepsilon^{-1}) & d = 2 \end{cases}$$

Which tells us that the typical size of $Y_t(x)$ is $\varepsilon^{2-d} \to \infty$. The estimates from last week are useless in this limit, because they depend on $L^p(\Lambda_\varepsilon)$ norms of Y_t.

This is a problem of small scales. It hints to the fact that Y^ε is not converging to a function on $\mathbb{T}^d \approx [0, 1]^d$, not even locally.

For convenience we define $\mathbb{T}_\varepsilon^d = \Lambda_{\varepsilon,1} = (\varepsilon \mathbb{Z} \cap [-1/2, 1/2))^d$.

5.1 Besov spaces

One way to deal with this problem and analyze what is going on in the equation (10) is to split all our functions in "blocks" which are nice.

This is accomplished via Littlewood–Paley decomposition, i.e. a nice partition of unity in Fourier space. We split every function $f: \mathbb{T}_\varepsilon^d \to \mathbb{R}$ in very nice pieces $(\Delta_i f)_{i \geqslant -1}$ as follows

$$f(x) = \sum_{i \geqslant -1} (\Delta_i f)(x),$$

where

$$\Delta_i f(x) := \sum_{k \in (\mathbb{Z} \cap [-\varepsilon^{-1}, \varepsilon^{-1}))^d} \rho(2^{-i} k) \hat{f}(k) e^{2\pi i k \cdot x} \qquad i \geqslant 0,$$

and

$$\Delta_{-1} f(x) := \sum_{k \in (\mathbb{Z} \cap [-\varepsilon^{-1}, \varepsilon^{-1}))^d} \chi(k) \hat{f}(k) e^{2\pi i k \cdot x},$$

where $\rho: \mathbb{R}^d \to \mathbb{R}_{\geqslant 0}$ and is such that

$$\chi(k) + \sum_{i \geqslant 0} \rho(2^{-i} k) = 1,$$

for a nice function $\chi: \mathbb{R}^d \to \mathbb{R}_{\geqslant 0}$ with support in a ball $\mathscr{B}$ of radius ≈ 1 around $k = 0 \in \mathbb{R}^d$ and ρ is supported on an annulus $\mathscr{A}$ of radius ≈ 1. All these functions are smooth (and some other properties we don't care about right now).

Therefore the Fourier transform of the LP block $\Delta_i f$ is supported on an annulus of size 2^i and that of $\Delta_{-1} f$ in a ball of radius 1.

Remark. For $\varepsilon > 0$ we have $\Delta_i f = 0$ if $2^i \gtrsim \varepsilon^{-1}$, so we sum over i up to $\approx \log_2 \varepsilon^{-1}$. Let us define N_ε to be this bound. So

$$f = \sum_{i=-1}^{N_\varepsilon} (\Delta_i f),$$

there is a technical subtlety here on how one handles the last last block but we will ignore it.

Now:

$$\mathbb{E}[(\Delta_i Y_t(x))^2] = \sum_{k \in (\mathbb{Z} \cap [-\varepsilon^{-1}, \varepsilon^{-1}))^d} \frac{\rho(2^{-i} k)^2}{(m^2 + \sum_i (2\varepsilon^{-1} \sin(\pi \varepsilon k_i))^2)}$$

$$\approx \sum_{k \in 2^i \mathscr{A} \subseteq (\mathbb{Z} \cap [-\varepsilon^{-1}, \varepsilon^{-1}))^d} \frac{\rho(2^{-i} k)^2}{(m^2 + |k|^2)} \approx \sum_{k \in 2^i \mathscr{A} \subseteq (\mathbb{Z} \cap [-\varepsilon^{-1}, \varepsilon^{-1}))^d} \frac{\rho(2^{-i} k)^2}{\underbrace{|k|^2}_{\approx (2^i)^2}} \lesssim (2^i)^{d-2}$$

Which says that $\Delta_i Y_t \approx (2^i)^{(d-2)/2}$ which is uniform in ε! (but of course not in i, and i can be large)

One can prove actually that $\Delta_i Y^\varepsilon$ converges to a nice C^∞ random function on $\mathbb{T}^d$ as $\varepsilon \to 0$.

This decomposition shift the problem of dealing with distribution to a problem of dealing with large sums.

Definition 4. *Let $\alpha \in \mathbb{R}$ and $p, q \in [1, +\infty]$. We say that $f \in B^{\alpha}_{p,q}$ (a Besov space) iff*

$$\|f\|_{B^{\alpha}_{p,q}} := \|i \geqslant -1 \mapsto 2^{i\alpha}\|\Delta_i f\|_{L^p}\|_{\ell^q} = \left[\sum_i (2^{i\alpha}\|\Delta_i f\|_{L^p})^q\right]^{1/q} < \infty.$$

These are Banach spaces.

In particular we will use $\mathscr{C}^{\alpha} = B^{\alpha}_{\infty,\infty}$ with norm $\|f\|_{\mathscr{C}^{\alpha}}$ such that

$$\|\Delta_i f\|_{L^\infty} \leqslant 2^{-\alpha i}\|f\|_{\mathscr{C}^{\alpha}}.$$

When $\alpha > 0$ there are spaces of regular functions, when $\alpha < 0$ these are just distributions (of course when $\varepsilon = 0$).

Moreover note that

$$\|f\|^2_{B^{\alpha}_{2,2}} = \sum_{i \geqslant -1} 2^{2i\alpha}\|\Delta_i f\|^2_{L^2(\mathbb{T}^d_\varepsilon)} = \varepsilon^d \sum_{x \in \mathbb{T}^d_\varepsilon} \sum_{i \geqslant -1} 2^{2i\alpha}|\Delta_i f(x)|^2 \approx \varepsilon^d \sum_{x \in \mathbb{T}^d_\varepsilon} |(1 - \Delta_\varepsilon)^{\alpha/2} f|^2 =: \|f\|^2_{H^\alpha}.$$

From the estimate above on Y one can prove that almost surely for any $t \in \mathbb{R}$, $Y^\varepsilon_t \in \mathscr{C}^{(d-2)/2-\kappa}$ for any small $\kappa > 0$ uniformly in ε. One can also prove that as a function of t is continuous and actually uniformly in ε

$$Y^\varepsilon \in C(\mathbb{R}; \mathscr{C}^{(d-2)/2-\kappa})$$

almost surely.

In the following κ will always denote an arbitrary small positive quantity. (this is a small loss of regularity due to the fact that we want almost sure statements).

Note that when $d = 2$ we have $Y^\varepsilon \in C(\mathbb{R}; \mathscr{C}^{-\kappa})$ and when $d = 3$ $Y^\varepsilon \in C(\mathbb{R}; \mathscr{C}^{-1/2-\kappa})$.

Take $f \in \mathscr{C}^\alpha$, $g \in \mathscr{C}^\beta$ then

$$fg = \sum_i \Delta_i f \sum_j \Delta_j g = \sum_{i,j} \Delta_i f \Delta_j g$$

to give a sense to this product one has to control the two (large) sums. The good way to do it is to split it in three pieces:

$$fg = \sum_{i,j} \Delta_i f \Delta_j g = \sum_{i < j-K} \Delta_i f \Delta_j g + \sum_{i > j+K} \Delta_i f \Delta_j g + \sum_{|i-j| \lesssim K} \Delta_i f \Delta_j g$$

$$=: (f \prec g) + (f \succ g) + (f \circ g)$$

and call them the paraproducts $(f \prec g)$, $(f \succ g) = (g \prec f)$ and the resonant term $(f \circ g)$.

Theorem 5. *The paraproducts are always well defined and*

$$\|f \prec g\|_{\mathscr{C}^\beta} \lesssim \|f\|_{\mathscr{C}^\alpha} \|g\|_{\mathscr{C}^\beta}, \qquad \alpha > 0,$$

$$\|f \prec g\|_{\mathscr{C}^{\alpha+\beta}} \lesssim \|f\|_{\mathscr{C}^\alpha} \|g\|_{\mathscr{C}^\beta}, \qquad \alpha < 0.$$

The resonant product is well-defined only if $\underline{\alpha + \beta > 0}$ *and in this case*

$$\|f \circ g\|_{\mathscr{C}^{\alpha+\beta}} \lesssim \|f\|_{\mathscr{C}^\alpha} \|g\|_{\mathscr{C}^\beta}.$$

Therefore fg *is well defined (and continuous) if* $\alpha + \beta > 0$ *and in this case*

$$\|fg\| \lesssim \|f\|_{\mathscr{C}^\alpha} \|g\|_{\mathscr{C}^\beta}.$$

We are allowed to multiply things only if regularity is ok, and the problem is in the resonant term.

5.2 Renormalization

Let's go back with these tools to our equation (10). In the r.h.s. we have

$$V'(Y+Z) = \lambda Y^3 + 3\lambda Y^2 Z + 3\lambda Y Z^2 + \lambda Z^3 + \beta Y + \beta Z.$$

By the product theorem we see that Y^3 is problematic since the regularity $\alpha = (2-d)/2 - \kappa$ of Y is negative. However Y is explicit, and we can do a probabilistic computation to prove that Y^3 converge as $\varepsilon \to 0$ to a well defined distribution provided it is *renormalized*.

Theorem 6. *There exists a constant* c_ε *such that the random field (renormalized square)*

$$\mathbb{Y}_t^{\varepsilon,2}(x) := (Y_t^\varepsilon(x))^2 - c_\varepsilon,$$

converges (in law) as $\varepsilon \to 0$ *to a random field* $\mathbb{Y}^2$ *in* $C(\mathbb{R}; \mathscr{C}^{2\alpha})$ *with* $\alpha = (2-d)/2 - \kappa < 0$ *(if* $d \geqslant 2$*).*

Similarly if $d = 2$ *the renormalized cube*

$$\mathbb{Y}_t^{\varepsilon,3}(x) := (Y_t^\varepsilon(x))^3 - 3 c_\varepsilon Y_t^\varepsilon(x),$$

converges as $\varepsilon \to 0$ *to a random field in* $C(\mathbb{R}; \mathscr{C}^{3\alpha})$ *while if* $d = 3$ *then convergence holds* $C^{-\kappa}(\mathbb{R}; \mathscr{C}^{3\alpha})$ *(where* $C^{-\kappa}$ *is a space of distributions in the time variable with negative regularity).*

Moreover one can take

$$c_\varepsilon := \mathbb{E}[(Y_t^\varepsilon(x))^2] \approx \varepsilon^{(2-d)}.$$

With this choice the renormalization corresponds to "Wick ordering".

Now we see that replacing

$$\beta = \beta_\varepsilon = \beta' - 3\lambda c_\varepsilon,$$

on has (with $\mathbb{Y}^1 = Y$)

$$V'(Y+Z) = \lambda \underbrace{(Y^3 - 3 c_\varepsilon Y)}_{\mathbb{Y}^3} + 3\lambda \underbrace{(Y^2 - c_\varepsilon)}_{\mathbb{Y}^2} Z + 3\lambda Y Z^2 + \lambda Z^3 + \beta' Y + \beta' Z$$

$$= \lambda \mathbb{Y}^3 + 3\lambda \mathbb{Y}^2 Z + 3\lambda \mathbb{Y}^1 Z^2 + \lambda Z^3 + \beta' \mathbb{Y}^1 + \beta' Z.$$

Magic: one constant works for both problematic terms... (there are reasons for that, namely subcriticality of this model).

Next problems: the products

$$\underbrace{\mathbb{Y}^2}_{\mathscr{C}^{2\alpha}} Z, \qquad \underbrace{\mathbb{Y}^1}_{\mathscr{C}^{\alpha}} Z^2.$$

Let's try to get some estimates for Z: we test the equation (10) with Z and integrate in space $\mathbb{T}_\varepsilon^d$

$$\frac{1}{2}\frac{\partial}{\partial t}\int_{\mathbb{T}_\varepsilon^d} Z_t^2 + \int_{\mathbb{T}_\varepsilon^d}\left[|\nabla_\varepsilon Z_t|^2 + m^2 |Z_t|^2 + \frac{\lambda}{2}|Z_t|^4\right]$$

$$= -\frac{1}{2}\int_{\mathbb{T}_\varepsilon^d} [\lambda \mathbb{Y}^3 Z + 3\lambda \mathbb{Y}^2 Z^2 + 3\lambda \mathbb{Y}^1 Z^3 + \beta' \mathbb{Y}^1 Z + \beta' Z^2].$$

The l.h.s tells me that I have control of the L^2, L^4 norm of Z but also of the H^1 norm of Z, this means we have some regularity for Z.

Note that we define:

$$\int_{\mathbb{T}_\varepsilon^d} \mathrm{d}x := \varepsilon^d \sum_{x \in \mathbb{T}_\varepsilon^d} .$$

Also remark that $H^1 = B_{2,2}^1$. In the Besov scale we have Sobolev spaces. The theory of products and paraproducts extends naturally to Besov space with indexes p, q other than ∞, ∞.

When $d = 2$ we have that $\mathbb{Y}^k \in \mathscr{C}^{-k\alpha}$ with $k = 1, 2, 3$ and $\alpha = -\kappa$ a small negative quantity. Therefore all the products in the a-priori r.h.s. are well defined assuming $Z \in H^1$ (the sums of regularities is positive!). For example one has estimates like (for some small δ and some large K)

$$\left|\int_{\mathbb{T}_\varepsilon^d} \mathbb{Y}^3 Z\right| \lesssim \|\mathbb{Y}^3\|_{\mathscr{C}^{3\alpha}} \|Z\|_{B_{1,1}^{4\kappa}} \lesssim C_\delta \|\mathbb{Y}^3\|_{\mathscr{C}^{3\alpha}}^K + \delta \|\nabla Z\|_{L^2}^2 + \delta \|Z\|_{L^2}^2,$$

$$\left|\int_{\mathbb{T}_\varepsilon^d} \mathbb{Y}^2 Z^2\right| \lesssim \|\mathbb{Y}^2\|_{\mathscr{C}^{2\alpha}} \|Z^2\|_{B_{1,1}^{3\kappa}} \lesssim C_\delta \|\mathbb{Y}^2\|_{\mathscr{C}^{2\alpha}}^K + \delta \|\nabla Z\|_{L^2}^2 + \delta \|Z\|_{L^4}^4,$$

$$\left|\int_{\mathbb{T}_\varepsilon^d} \mathbb{Y}^1 Z^3\right| \lesssim \|\mathbb{Y}^1\|_{\mathscr{C}^{\alpha}} \|Z^3\|_{B_{1,1}^{2\kappa}} \lesssim C_\delta \|\mathbb{Y}^1\|_{\mathscr{C}^{\alpha}}^K + \delta \|\nabla Z\|_{L^2}^2 + \delta \|Z\|_{L^4}^4,$$

So overall we can obtain (via PDE methods only, no probability here)

$$\frac{1}{2}\frac{\partial}{\partial t}\int_{\mathbb{T}_\varepsilon^d} Z_t^2 + (1-\delta)\int_{\mathbb{T}_\varepsilon^d}\left[|\nabla_\varepsilon Z_t|^2 + m^2 |Z_t|^2 + \frac{\lambda}{2}|Z_t|^4\right] \leqslant Q(\mathbb{Y}_t^\varepsilon),$$

where

$$Q(\mathbb{Y}_t^\varepsilon) := 1 + C \sum_{k=1,2,3} \|\mathbb{Y}_t^k\|_{\mathscr{C}^{k\alpha}}^K,$$

for some power K. This estimate holds for all paths of Y since note that $Y^\varepsilon \in C\big(\mathbb{R} \times \mathbb{R}^{\mathbb{T}_\varepsilon^d}; \mathbb{R}\big)$ so it is clear that $Q(\mathbb{Y}_t^\varepsilon) < \infty$.

The real problem is: what happens when $\varepsilon \to 0$?

Remember that we constructed a stationary coupling $\mathbb{P}^\varepsilon$ such that under $\mathbb{P}^\varepsilon$ the processes Y and Z are stationary and

$$X = Y + Z$$

is also stationary and such that $X_t \sim \nu^\varepsilon$ (recall $M = 1$ here).

Under this coupling this estimate implies that

$$\underbrace{\frac{1}{2}\frac{\partial}{\partial t}\mathbb{E}\int_{\mathbb{T}_\varepsilon^d} Z_t^2}_{=0 \text{ (by stationarity)}} + \mathbb{E}\int_{\mathbb{T}_\varepsilon^d}\left[|\nabla_\varepsilon Z_t|^2 + m^2|Z_t|^2 + \frac{\lambda}{2}|Z_t|^4\right] \lesssim \mathbb{E}\, Q(\mathbb{Y}_t^\varepsilon) = \mathbb{E}\, Q(\mathbb{Y}_0^\varepsilon),$$

but again, using some probability theory one can prove that

$$\sup_\varepsilon \mathbb{E}\, Q(\mathbb{Y}_0^\varepsilon) < \infty,$$

since again Y is well known and estimates are relatively easy. As a result one obtain uniform estimates of the form

$$\sup_\varepsilon \mathbb{E}\int_{\mathbb{T}_\varepsilon^d}\left[|\nabla_\varepsilon Z_0|^2 + m^2|Z_0|^2 + \frac{\lambda}{2}|Z_0|^4\right] < \infty.$$

This estimate is a key point because from that one can derive tightness of the family $(\nu^\varepsilon)_{\varepsilon>0}$. Indeed let γ^ε the law of (Y_0, Z_0) under $\mathbb{P}^\varepsilon$, we have

$$\sup_\varepsilon \int (\|\psi\|_{\mathscr{C}^{-\alpha}}^2 + \|\nabla\zeta\|_{L^2}^2 + \|\zeta\|_{L^2}^2 + \|\zeta\|_{L^4}^4)\,\gamma^\varepsilon(\mathrm{d}\psi \times \mathrm{d}\zeta)$$

$$= \sup_\varepsilon \mathbb{E}_{\mathbb{P}^\varepsilon}[\|Y_0\|_{\mathscr{C}^{-\alpha}}^2 + \|\nabla Z_0\|_{L^2}^2 + \|Z_0\|_{L^2}^2 + \|Z_0\|_{L^4}^4] \lesssim \sup_\varepsilon \mathbb{E}\, Q(\mathbb{Y}_0^\varepsilon) < \infty,$$

note that $\|Y_0\|_{\mathscr{C}^{-\alpha}}^2 \lesssim Q(\mathbb{Y}_0^\varepsilon)$ if K large enough.

This gives tightness of $(\gamma^\varepsilon)_{\varepsilon\geqslant 0}$ in $\mathscr{C}^{-2\alpha} \times (H^{1-\kappa} \cap L^4)$ (some loss of regularity to guarantee the required compactness). Projecting down to $(\nu^\varepsilon)_\varepsilon$ (taking the sum of the two factors) one get tightness of $(\nu^\varepsilon)_\varepsilon$ in $H^{-2\alpha} = B_{2,2}^{-2\alpha}$:

$$\int \|\varphi\|_{B_{2,2}^{-\alpha}}^2 \nu^\varepsilon(\mathrm{d}\varphi) = \int \|\psi + \zeta\|_{B_{2,2}^{-\alpha}}^2 \gamma^\varepsilon(\mathrm{d}\psi \times \mathrm{d}\zeta) \leqslant 2\int (\|\psi\|_{B_{2,2}^{-\alpha}}^2 + \|\zeta\|_{B_{2,2}^{-\alpha}}^2)\,\gamma^\varepsilon(\mathrm{d}\psi \times \mathrm{d}\zeta)$$

$$\leqslant 2\int (\|\psi\|_{\mathscr{C}^{-\alpha}}^2 + \|\zeta\|_{H^1}^2)\,\gamma^\varepsilon(\mathrm{d}\psi \times \mathrm{d}\zeta),$$

which is uniformly bounded in ε. So we can extract an accumulation point ν (a measure on $H^{-2\alpha}(\mathbb{T}^2)$.

Theorem 7. *Provided $d=2$ and we take $\beta=-3\lambda c_\varepsilon+\beta'$ for some constant $\beta'\in\mathbb{R}$ and $c_\varepsilon=\mathbb{E}[Y_t^\varepsilon(x)^2]$ then the family $(\nu_\varepsilon)_\varepsilon$ is tight in $H^{-2\alpha}(\mathbb{T}^2)$.*

6 UV limit in three dimensions

What happens in $d=3$. Let us go back to the a-priori estimates: test the equation (10) where $V'(\varphi)=\lambda\varphi^3+\beta\varphi$ (and as in two dimensions we take $\beta=-3\lambda c_\varepsilon+\beta'$) with Z and integrate in space $\mathbb{T}_\varepsilon^3$:

$$\frac{1}{2}\frac{\partial}{\partial t}\int_{\mathbb{T}_\varepsilon^3} Z_t^2+\int_{\mathbb{T}_\varepsilon^3}\left[|\nabla_\varepsilon Z_t|^2+m^2|Z_t|^2+\frac{\lambda}{2}|Z_t|^4\right]$$

$$=-\frac{1}{2}\int_{\mathbb{T}_\varepsilon^3}[\lambda\,\mathbb{Y}^3 Z+3\lambda\,\mathbb{Y}^2 Z^2+3\lambda\,\mathbb{Y}^1 Z^3+\beta'\,\mathbb{Y}^1 Z+\beta' Z^2].$$

But now $\mathbb{Y}^2$ has regularity $-1-2\kappa$ and $\mathbb{Y}^3$ even worser than $-3/2-3\kappa$. For Z we can hope only for H^1 regularity from these estimates. *Big problem!!*

The term $\mathbb{Y}^1 Z^3$, $\mathbb{Y}^1 Z$ are still ok because $\mathbb{Y}^1$ has regularity $-1/2-\kappa$.

We go back to the equation (10) and write it more explicitly

$$\frac{\partial}{\partial t}Z_t+(m^2-\Delta_\varepsilon)Z_t=\underbrace{-\frac{1}{2}\lambda\,\mathbb{Y}^3}_{\mathscr{C}^{-3/2-3\kappa}}-\frac{3}{2}\lambda\,\mathbb{Y}^2 Z+\cdots$$

From the theory of parabolic equations one sees that Z cannot have better regularity that $2+-3/2-3\kappa=1/2-3\kappa>0$ surely it cannot be H^1. Moreover in this case we even have a worser problem for the term $\mathbb{Y}^2 Z$ which is a prod. of something of reg. $-1-2\kappa$ and something of reg. $1/2-3\kappa$ which do not sum up to a positive quantity. The first step is to separated the problems in the product $\mathbb{Y}^2 Z$ via a decomposition, we write

$$\mathbb{Y}^2 Z=\mathbb{Y}^2\succ Z+\mathbb{Y}^2\preccurlyeq Z,$$

where $\mathbb{Y}^2\preccurlyeq Z=\mathbb{Y}^2\prec Z+\mathbb{Y}^2\circ Z$. By paraproducts estimates one has that $\mathbb{Y}^2\succ Z$ has regularity of $\mathbb{Y}^2$ that is $-1-2\kappa$ and it is well-defined. The term containing the resonant product $\mathbb{Y}^2\preccurlyeq Z$ is however not well defined.

Define a new stochastic object $\mathbb{Y}^{[3],\varepsilon}$ to be the solution of the equation

$$\frac{\partial}{\partial t}\mathbb{Y}_t^{[3],\varepsilon}+(m^2-\Delta_\varepsilon)\,\mathbb{Y}_t^{[3],\varepsilon}=-\frac{1}{2}\lambda\,\mathbb{Y}_t^{3,\varepsilon},$$

(for example, take the stationary solution). Again this is a very explicit functional of the Gaussian process Y^ε and will be easy to analyze, in particular one can show that uniformly in ε it belongs to

$$\mathbb{Y}^{[3],\varepsilon}\in C(\mathbb{R},\mathscr{C}^{1/2-3\kappa}(\mathbb{T}^3)),$$

in the sense that, for example,

$$\sup_{\varepsilon} \mathbb{E}\left[\sup_{t\in[0,T]} \|\mathbb{Y}_t^{[3],\varepsilon}\|^K_{\mathscr{C}^{1/2-3\kappa}(\mathbb{T}^3)}\right] < \infty,$$

for any K, T.

Now define $\mathbb{H}$ as the solution to

$$\frac{\partial}{\partial t}\mathbb{H}_t + (m^2 - \Delta_\varepsilon)\,\mathbb{H}_t = -\frac{1}{2}\lambda\,\mathbb{Y}_t^3 - \frac{3}{2}\lambda\,\mathbb{Y}_t^2 \succ \mathbb{H}_t, \tag{11}$$

this is a linear equation which can be easily solved and analyzed and its solution $\mathbb{H}$ does not looks much different than $\mathbb{Y}^{[3],\varepsilon}$ and lives also in $C(\mathbb{R}, \mathscr{C}^{1/2-3\kappa}(\mathbb{T}^3))$.

Define Φ as

$$Z =: \mathbb{H} + \Phi$$

which solves

$$\begin{aligned}\frac{\partial}{\partial t}\Phi_t &= (\Delta_\varepsilon - m^2)\,\Phi_t - \frac{\lambda}{2}[-3\lambda\,\mathbb{Y}^2 \succ \mathbb{H} + 3\,\mathbb{Y}^2 \succ Z + 3\,\mathbb{Y}^2 \circ Z + 3\,\mathbb{Y}^2 \prec Z + 3\,\mathbb{Y}Z^2 + Z^3] \\ &= (\Delta_\varepsilon - m^2)\,\Phi_t - \frac{\lambda}{2}\left[3\,\mathbb{Y}^2 \succ \Phi + \underbrace{3\,\mathbb{Y}^2 \circ \Phi + 3\,\mathbb{Y}^2 \circ \mathbb{H}}_{\text{dangerous terms!!!}} + 3\,\mathbb{Y}^2 \prec Z + 3\,\mathbb{Y}Z^2 + Z^3\right]\end{aligned} \tag{12}$$

This is the right equation to get a-priori estimates for (almost since Φ cannot be expected to be in H^1 exactly due to this equation). Let's test it with Φ to get

$$\frac{1}{2}\frac{\partial}{\partial t}\int_{\mathbb{T}_\varepsilon^3}\Phi^2 + \int_{\mathbb{T}_\varepsilon^3}\Phi\,(m^2 - \Delta_\varepsilon)\,\Phi + \frac{\lambda}{2}\int_{\mathbb{T}_\varepsilon^3}\Phi^4$$

$$= \int_{\mathbb{T}_\varepsilon^3}\Phi\left[-\frac{3}{2}\lambda\,\mathbb{Y}^2 \succ \Phi - \frac{3}{2}\lambda\,\mathbb{Y}^2 \circ \Phi - \frac{3}{2}\lambda\,\mathbb{Y}^2 \circ \mathbb{H} - \frac{1}{2}\beta'(\mathbb{Y}^1 + Z)\right]$$

$$+ \int_{\mathbb{T}_\varepsilon^3}\Phi\left[-\frac{3}{2}\lambda\,\mathbb{Y}^2 \prec Z_t - \frac{3}{2}\lambda\,\mathbb{Y}^1 Z^2\right] - \frac{\lambda}{2}\int_{\mathbb{T}_\varepsilon^3}\Phi((\mathbb{H} + \Phi)^3 - \Phi^3)$$

We have now to cross fingers and check that all the terms in the r.h.s. can be controlled with the l.h.s.

The term

$$-\frac{\lambda}{2}\int_{\mathbb{T}_\varepsilon^3}\Phi_t((\mathbb{H}_t + \Phi_t)^3 - \Phi_t^3),$$

is not scary at all since $\mathbb{H}$ is a nice function and it contains only powers less than 4 of Φ so it can be controlled via the $\int\Phi^4$ is the l.h.s. (like in the infinite vol estimates of last week). The term

$$-\frac{3}{2}\lambda\int_{\mathbb{T}_\varepsilon^3}\Phi_t\,\mathbb{Y}^1 Z_t^2 = -\frac{3}{2}\lambda\int_{\mathbb{T}_\varepsilon^3}\Phi_t\,\mathbb{Y}^1(\mathbb{H} + \Phi_t)^2$$

is also fine since $\mathbb{Y}^1$ is only $-1/2-\kappa$ irregular and we have the H^1 norm of Φ and it is at most cubic in Φ^3. With some work one can get a nice estimate. Note however that this term will contain products

$$\mathbb{Y}^1 \mathbb{H}, \quad \mathbb{Y}^1 \mathbb{H}^2,$$

which are not well defined because $\mathbb{H}$ is only of regularity $1/2-2\kappa$ so the reg. do not sum up to positive number. However these terms can be analyzed with probabilistic estimates and shown to be well defined and not needing renormalization. We will assume in the following that they have uniform estimates as $\varepsilon \to 0$ in

$$\mathbb{Y}^1 \mathbb{H}, \mathbb{Y}^1 \mathbb{H}^2 \in C(\mathbb{R}; \mathscr{C}^{-1/2-\kappa}(\mathbb{T}^3)).$$

We are worried about the terms:

$$-\frac{3}{2} \lambda \int_{\mathbb{T}_\varepsilon^3} \Phi_t [\mathbb{Y}^2 \succ \Phi_t + \mathbb{Y}^2 \preccurlyeq \Phi_t]$$

since Φ is not regular enough for $\mathbb{Y}^2$. Here we use the following fact.

Lemma 8. *We have*

$$D(f, g, h) := \int_{\mathbb{T}_\varepsilon^3} f(g \succ h) - \int_{\mathbb{T}_\varepsilon^3} (g \circ f) h$$

is well defined and continuous when the sum of the regularities of f, g, h is positive. For example

$$|D(f, g, h)| \leqslant \|f\|_{H^\alpha} \|h\|_{H^\gamma} \|g\|_{\mathscr{C}^\beta}$$

whenever $\alpha + \beta + \gamma > 0$.

Using this lemma we have

$$\int_{\mathbb{T}_\varepsilon^3} \Phi_t [\mathbb{Y}^2 \succ \Phi_t + \mathbb{Y}^2 \circ \Phi_t] = \int_{\mathbb{T}_\varepsilon^3} \Phi_t [2\, \mathbb{Y}^2 \succ \Phi_t] + D(\Phi_t, \mathbb{Y}_t^2, \Phi_t)$$

We got rid of the resonant product but the term

$$\int_{\mathbb{T}_\varepsilon^3} \Phi_t [2\, \mathbb{Y}^2 \succ \Phi_t]$$

is still dangerous.

Going back to the a-priori estimate we focus on two terms

$$\cdots + \int_{\mathbb{T}_\varepsilon^3} \Phi_t (m^2 - \Delta_\varepsilon) \Phi_t = -3\lambda \int_{\mathbb{T}_\varepsilon^3} \Phi_t [\mathbb{Y}^2 \succ \Phi_t] + \cdots$$

and try to cancel the one in r.h.s. using that in the l.h.s. This is possible by defining

$$\Psi_t := \Phi_t + \frac{3\lambda}{2} \mathcal{Q}^{-1}[\mathbb{Y}^2 \succ \Phi_t]$$

where

$$\mathcal{Q} := (m^2 - \Delta_\varepsilon).$$

Substituting the estimate (i.e. we are completing the above square). One get

$$\int_{\mathbb{T}_\varepsilon^3} \Phi_t \mathcal{Q} \Phi_t + 3\lambda \int_{\mathbb{T}_\varepsilon^3} \Phi_t[\mathbb{Y}^2 \succ \Phi_t]$$

$$= \int_{\mathbb{T}_\varepsilon^3} \left[\Psi_t - \frac{3\lambda}{2} \mathcal{Q}^{-1}[\mathbb{Y}^2 \succ \Phi_t]\right] \mathcal{Q} \left[\Psi_t - \frac{3\lambda}{2} \mathcal{Q}^{-1}[\mathbb{Y}^2 \succ \Phi_t]\right] + 3\lambda \int_{\mathbb{T}_\varepsilon^3} \Phi_t[\mathbb{Y}^2 \succ \Phi_t]$$

$$= \int_{\mathbb{T}_\varepsilon^3} \Psi_t \mathcal{Q} \Psi_t - 3\lambda \int_{\mathbb{T}_\varepsilon^3} \Psi_t[\mathbb{Y}^2 \succ \Phi_t] + \frac{9\lambda^2}{4} \int_{\mathbb{T}_\varepsilon^3} (\mathbb{Y}^2 \succ \Phi_t) \mathcal{Q}^{-1} (\mathbb{Y}^2 \succ \Phi_t) + 3\lambda \int_{\mathbb{T}_\varepsilon^3} \Phi_t[\mathbb{Y}^2 \succ \Phi_t]$$

$$= \int_{\mathbb{T}_\varepsilon^3} \Psi_t \mathcal{Q} \Psi_t - \frac{9\lambda^2}{4} \int_{\mathbb{T}_\varepsilon^3} (\mathbb{Y}^2 \succ \Phi_t) \mathcal{Q}^{-1} (\mathbb{Y}^2 \succ \Phi_t)$$

6.1 The paracontrolled a-priori estimate

To recap, we defined $\mathbb{H}$ to be

$$\frac{\partial}{\partial t} \mathbb{H}_t + (m^2 - \Delta_\varepsilon) \mathbb{H}_t = -\frac{\lambda}{2} \mathbb{Y}_t^3 - \frac{3\lambda}{2} \mathbb{Y}_t^2 \succ \mathbb{H}_t,$$

(solve this equation by a fix-point) and defined $\Phi := Z - \mathbb{H}$ which satisfies

$$\frac{\partial}{\partial t} \Phi_t = (\Delta_\varepsilon - m^2) \Phi_t - \frac{\lambda}{2}[-3\lambda \mathbb{Y}^2 \succ \mathbb{H} + 3\mathbb{Y}^2 \succ Z + 3\mathbb{Y}^2 \circ Z + 3\mathbb{Y}^2 \prec Z + 3\mathbb{Y} Z^2 + Z^3]$$

$$= (\Delta_\varepsilon - m^2) \Phi_t - \frac{\lambda}{2}\left[3\mathbb{Y}^2 \succ \Phi + \underbrace{3\mathbb{Y}^2 \circ \Phi + 3\mathbb{Y}^2 \circ \mathbb{H}}_{\text{dangerous terms!!!}} + 3\mathbb{Y}^2 \prec Z + 3\mathbb{Y} Z^2 + Z^3\right].$$

Recall the various regularities (we use κ for an arbitrary small >0 which can be different from line to line)

term	reg
$\mathbb{Y}^1$	$-1/2 - \kappa$
$\mathbb{Y}^2$	$-1 - \kappa$
$\mathbb{Y}^3$	"$-3/2 - \kappa$"(as space-time distribution)
$\mathbb{H}$	$1/2 - \kappa$
$\mathbb{Y}^2 \succ \Phi$	$-1 - \kappa$
Φ	$1 - \kappa$

Then we tested with Φ to get

$$\frac{1}{2}\frac{\partial}{\partial t}\int_{\mathbb{T}_\varepsilon^3}\Phi_t^2+\int_{\mathbb{T}_\varepsilon^3}\Phi_t(m^2-\Delta_\varepsilon)\,\Phi_t+\frac{\lambda}{2}\int_{\mathbb{T}_\varepsilon^3}\Phi_t^4$$

$$=\int_{\mathbb{T}_\varepsilon^3}\Phi\left[-\frac{3}{2}\lambda\,\mathbb{Y}^2\succ\Phi-\frac{3}{2}\lambda\,\mathbb{Y}^2\circ\Phi\right]$$

$$+\int_{\mathbb{T}_\varepsilon^3}\Phi\left[-\frac{3}{2}\lambda\,\mathbb{Y}^2\circ\mathbb{H}-\frac{1}{2}\beta'(\mathbb{Y}^1+Z)\right]$$

$$-\frac{3}{2}\lambda\int_{\mathbb{T}_\varepsilon^3}\Phi(\mathbb{Y}^2\prec Z+\mathbb{Y}^1Z^2)-\frac{\lambda}{2}\int_{\mathbb{T}_\varepsilon^3}\Phi((\mathbb{H}+\Phi)^3-\Phi^3)$$

and we did a transformation to the combination (in which all the terms are "ill defined", i.e. I cannot hope to control them separately in the limit)

$$A:=\int_{\mathbb{T}_\varepsilon^3}\Phi\,(m^2-\Delta_\varepsilon)\,\Phi+\int_{\mathbb{T}_\varepsilon^3}\Phi\left[\frac{3}{2}\lambda\,\mathbb{Y}^2\succ\Phi+\frac{3}{2}\lambda\,\mathbb{Y}^2\circ\Phi\right]$$

we use a "commutator lemma" to replace $\int\Phi(\mathbb{Y}^2\circ\Phi)$ with $\int(\mathbb{Y}^2\succ\Phi)\,\Phi$ modulo nice error term:

$$A=\int_{\mathbb{T}_\varepsilon^3}\Phi\,(m^2-\Delta_\varepsilon)\,\Phi+\int_{\mathbb{T}_\varepsilon^3}\Phi[3\,\lambda\,\mathbb{Y}^2\succ\Phi]+\lambda D(\Phi,\mathbb{Y}^2,\Phi)$$

Then we defined Ψ so that

$$\Phi=-\frac{3\,\lambda}{2}(m^2-\Delta_\varepsilon)^{-1}[\mathbb{Y}^2\succ\Phi]+\Psi,$$

$$A=\int_{\mathbb{T}_\varepsilon^3}\Psi_t(m^2-\Delta_\varepsilon)\,\Psi_t+\frac{9\,\lambda}{4}\int_{\mathbb{T}_\varepsilon^3}(\mathbb{Y}^2\succ\Phi_t)(m^2-\Delta_\varepsilon)^{-1}(\mathbb{Y}^2\succ\Phi_t)+\lambda D(\Phi,\mathbb{Y}^2,\Phi)$$

At this point we have decomposed X as

$$X=\mathbb{Y}^1+\mathbb{H}-\frac{3\,\lambda}{2}(m^2-\Delta_\varepsilon)^{-1}[\mathbb{Y}^2\succ\Phi]+\Psi \tag{13}$$

where

$$\Phi=X-\mathbb{Y}^1-\mathbb{H}=-\frac{3\,\lambda}{2}(m^2-\Delta_\varepsilon)^{-1}[\mathbb{Y}^2\succ\Phi]+\Psi$$

with these different functions satisfying the a-priori equation

$$\frac{1}{2}\frac{\partial}{\partial t}\int_{\mathbb{T}_\varepsilon^3}\Phi^2+\int_{\mathbb{T}_\varepsilon^3}\Psi\,(m^2-\Delta_\varepsilon)\,\Psi+\frac{\lambda}{2}\int_{\mathbb{T}_\varepsilon^3}\Phi^4$$

$$=\int_{\mathbb{T}_\varepsilon^3}\left[-\frac{9\,\lambda^2}{4}\int_{\mathbb{T}_\varepsilon^3}(\mathbb{Y}^2\succ\Phi_t)(m^2-\Delta_\varepsilon)^{-1}(\mathbb{Y}^2\succ\Phi_t)-\frac{3}{2}\lambda\,\Phi\,\mathbb{Y}^2\circ\mathbb{H}-\frac{1}{2}\beta'\,\Phi(\mathbb{Y}^1+Z)\right]$$

$$+\lambda D(\Phi,\mathbb{Y}^2,\Phi)-\frac{3}{2}\lambda\int_{\mathbb{T}_\varepsilon^3}\Phi(\mathbb{Y}^2\prec Z+\mathbb{Y}^1Z^2)-\frac{\lambda}{2}\int_{\mathbb{T}_\varepsilon^3}\Phi((\mathbb{H}+\Phi)^3-\Phi^3)$$

The good guys are on the l.h.s and the bad guys on the r.h.s., with the ugly guys in orange.

The terms in orange are still out of control, in particular they contain products which are not well defined (because the regularities do not sum up to positive).

Let us pause a moment and try to understand the meaning of the decomposition (13): this is the key point of these new approaches to singular SPDEs (i.e. regularity structures or paracontrolled distributions). The message is that we cannot just look at generic functions in a given vector space (like in classical PDE theory) but we need to specify the solution as an "expansion" in terms (explicit or implicit) of different character. In the paracontrolled approach this involves the regularity of the various terms

$$X = \underbrace{\mathbb{Y}^1}_{-1/2-\kappa} + \underbrace{\mathbb{H}}_{1/2-\kappa} \underbrace{- \frac{3\lambda}{2}(m^2 - \Delta_\varepsilon)^{-1}[\mathbb{Y}^2 \succ \Phi]}_{1-\kappa} + \underbrace{\Psi}_{H^1},$$

(actually Ψ is even better than H^1, if I remember correctly it has regularity $3/2$).

For example one could see from this that for the LP blocks one has

$$\begin{array}{ll} \Delta_i X & \sim (2^i)^{1/2+\kappa}, \\ \Delta_i X - \Delta_i \mathbb{Y}^1 & \sim (2^i)^{-1/2-\kappa} \\ \Delta_i X - \Delta_i \mathbb{Y}^1 - \Delta_i \mathbb{H} & \sim (2^i)^{-1-\kappa} \\ \Delta_i X - \Delta_i \mathbb{Y}^1 - \Delta_i \mathbb{H} + \frac{3\lambda}{2}\Delta_i\{(m^2 - \Delta_\varepsilon)^{-1}[\mathbb{Y}^2 \succ \Phi]\} & \sim (2^i)^{-1} \end{array}$$

which can be interpreted by saying that my solution lives in a very particular subspace of the space of Besov functions of regularity $-1/2$ (we could take for example $H^{-1/2-\kappa}$).

In particular the stochastic objects $\mathbb{Y}^1, \mathbb{H}, \mathbb{Y}^2$ do not have better regularity as those stated (i.e. they are almost surely not in $\mathscr{C}^{-1/2}, \mathscr{C}^{1/2}, \mathscr{C}^1$ (think about Hölder regularity of BM).

6.2 The second renormalization

We need to understand what is going on with the red term

$$\int_{\mathbb{T}_\varepsilon^3} \left[-\frac{9\lambda^2}{4} \int_{\mathbb{T}_\varepsilon^3} (\mathbb{Y}^2 \succ \Phi_t)(m^2 - \Delta_\varepsilon)^{-1}(\mathbb{Y}^2 \succ \Phi_t) - \frac{3}{2}\lambda \Phi \mathbb{Y}^2 \circ \mathbb{H} - \frac{1}{2}\beta' \Phi(\mathbb{Y}^1 + Z) \right]$$

which contains not-well defined products.

Start with $\mathbb{Y}^2 \circ \mathbb{H}$: use the definition of $\mathbb{H}$ (where $\mathscr{L} = \partial_t + (m^2 - \Delta_\varepsilon)$)

$$\mathbb{H} = -\frac{\lambda}{2} \mathbb{Y}^{[3]} - \frac{3\lambda}{2} \mathscr{L}^{-1}(\mathbb{Y}^2 \succ \mathbb{H}),$$

recall also that $\mathbb{Y}^{[3]} = \mathscr{L}^{-1} \mathbb{Y}^3$ (with reg. $1/2 - \kappa$), and write it as

$$\mathbb{Y}^2 \circ \mathbb{H} = -\frac{\lambda}{2} \mathbb{Y}^2 \circ \mathbb{Y}^{[3]} - \frac{3\lambda}{2} \mathbb{Y}^2 \circ \mathscr{L}^{-1}(\mathbb{Y}^2 \succ \mathbb{H}).$$

For $\mathbb{Y}^2 \circ \mathbb{Y}^{[3]}$ we can show by probabilistic arguments involving Wick products (i.e. explicit formulas for polynomials of Gaussian) that one can define other polynomials $\mathbb{Y}^{2\circ[3]}$ and $\mathbb{Y}^{2\circ[2]}$

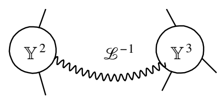
$$\mathbb{Y}^2 \circ \mathbb{Y}^{[3]} = \mathbb{Y}^2 \circ \mathscr{L}^{-1}\,\mathbb{Y}^3 = [\![Y^2]\!] \circ \mathscr{L}^{-1}[\![Y^3]\!] = \mathbb{Y}^{2\circ[3]} + 3\,d_\varepsilon\,\mathbb{Y}^1,$$

$$\mathbb{Y}^2 \circ \mathscr{L}^{-1}\,\mathbb{Y}^2 = \mathbb{Y}^{2\circ[2]} + d_\varepsilon$$

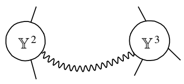
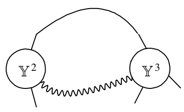
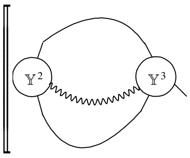
where d_ε is a constant which diverges logarithmically with ε. This is not much different from what we did in $d=2$ and in $d=3$ for the products Y^3, Y^2. The random field $\mathbb{Y}^{2\circ[3]}$ and $\mathbb{Y}^{2\circ[2]}$ converge as $\varepsilon \to 0$ to well defined random field such that

$$\boxed{\begin{gathered}\mathbb{Y}^{2\circ[3]} := \mathbb{Y}^2 \circ \mathbb{Y}^{[3]} - 3\,d_\varepsilon\,\mathbb{Y}^1 \in C(\mathbb{R}_+, \mathscr{C}^{1/2-\kappa}) \\ \mathbb{Y}^{2\circ[2]} := \mathbb{Y}^2 \circ \mathscr{L}^{-1}\,\mathbb{Y}^2 - d_\varepsilon \in C(\mathbb{R}_+, \mathscr{C}^{-\kappa})\end{gathered}}$$

In terms of Feynman graphs one could write

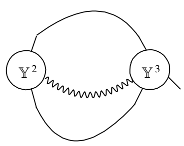
$\mathbb{Y}^2 \circ \mathbb{Y}^{[3]} =$ 

which can be decomposed in orthogonal terms

$=$ [[]] $+ 3\,2$ [[]] $+ 3\,2^2$ [[]]

$$= [\![Y^2]\!] \diamond_0 \mathscr{L}^{-1}[\![Y^3]\!] + 3\,2[\![Y^2]\!] \diamond_1 \mathscr{L}^{-1}[\![Y^3]\!] + 3\,2^2[\![Y^2]\!] \diamond_2 \mathscr{L}^{-1}[\![Y^3]\!]$$

and one has that the last one is diverging while the other two are well defined

[[]] $\approx \int_{\mathbb{T}_\varepsilon^3} \mathrm{d}x \underbrace{P(x-y)}_{\mathscr{L}^{-1}} \underbrace{G(x-y)^2}_{\text{two contraction lines}} Y(y) \approx \underbrace{\int_{\mathbb{T}_\varepsilon^3} \mathrm{d}x \frac{1}{|x|^3}}_{\propto d_\varepsilon} Y(y)$

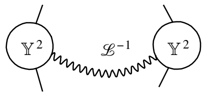
since the correlation function

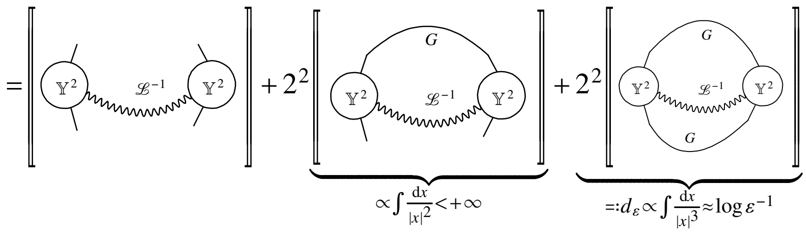
$$G(x-y) = \mathbb{E}[Y(x)Y(y)] \approx \sum_{k \in \mathbb{Z}^3 \cap [-\varepsilon^{-1}, \varepsilon^{-1}]^3} \frac{e^{ik(x-y)}}{k^2 + m^2} \approx \frac{1}{|x-y|}$$

and the kernel P of $\mathscr{L}^{-1}$ behaves in the same way

$$P(x-y) \approx |x-y|^{-1}.$$

For $\mathbb{Y}^2 \circ \mathscr{L}^{-1}\,\mathbb{Y}^2$ one can do the same:

$\mathbb{Y}^2 \circ \mathbb{Y}^{[2]} =$ 



$$= \mathbb{Y}^{2\circ[2]} + d_\varepsilon$$

where $\mathbb{Y}^{2\circ[2]}$ denotes the sum of the first two graphs.

Let us go back to $\mathbb{Y}^2 \circ \mathbb{H}$. The first step is to use commutator lemmas for paraproducts and resonant products:

Commutator lemmas roughly say that one can usually write

$$f \circ (g \succ h) \approx (f \circ g) h$$

modulo "nice terms". Similar statements can be made when there are other nice linear operations in between, e.g.

$$f \circ \mathscr{L}^{-1}(g \succ h) \approx (f \circ \mathscr{L}^{-1} g) h, \qquad f \circ (m^2 - \Delta)^{-1}(g \succ h) \approx (f \circ (m^2 - \Delta)^{-1} g) h$$

This is enough to show that

$$\mathbb{Y}^2 \circ \mathscr{L}^{-1}(\mathbb{Y}^2 \succ \mathbb{H}) = \underbrace{[\mathbb{Y}^2 \circ \mathscr{L}^{-1}(\mathbb{Y}^2)]}_{\text{ugly guy!}}\mathbb{H} + \underbrace{C(\mathbb{Y}^2, \mathbb{Y}^2, \mathbb{H})}_{\text{nice commutator}}$$

$$= (\mathbb{Y}^{2\circ[2]} + d_\varepsilon)\,\mathbb{H} + C(\mathbb{Y}^2, \mathbb{Y}^2, \mathbb{H})$$

Therefore we can handle the full term $\mathbb{Y}^2 \circ \mathbb{H}$ as

$$\mathbb{Y}^2 \circ \mathbb{H} = -\frac{\lambda}{2} \underbrace{\mathbb{Y}^2 \circ \mathbb{Y}^{[3]}}_{\text{uglyguy!}} - \frac{3\lambda}{2}\,\mathbb{Y}^2 \circ \mathscr{L}^{-1}(\mathbb{Y}^2 \succ \mathbb{H})$$

$$= -\frac{\lambda}{2}(\mathbb{Y}^{2\circ[3]} + 3 d_\varepsilon\,\mathbb{Y}^1) - \frac{3\lambda}{2}(\mathbb{Y}^{2\circ[2]} + d_\varepsilon)\,\mathbb{H} - \frac{3\lambda}{2} C(\mathbb{Y}^2, \mathbb{Y}^2, \mathbb{H})$$

$$= -\frac{\lambda}{2}\,\mathbb{Y}^{2\circ[3]} - \frac{3\lambda}{2}\,\mathbb{Y}^{2\circ[2]}\,\mathbb{H} - \frac{3\lambda}{2} C(\mathbb{Y}^2, \mathbb{Y}^2, \mathbb{H}) - \frac{3\lambda}{2} d_\varepsilon(\mathbb{Y}^1 + \mathbb{H})$$

and we see precisely how $\mathbb{Y}^2 \circ \mathbb{H}$ diverges as $\varepsilon \to 0$, due to the presence of d_ε.

Now our task is to handle the other dangerous term (highlighted in red):

$$\int_{\mathbb{T}_\varepsilon^3} (\mathbb{Y}^2 \succ \Phi_t)\, \mathcal{Q}^{-1} (\mathbb{Y}^2 \succ \Phi_t)$$

with $\mathcal{Q} = (m^2 - \Delta_\varepsilon)$. We can decompose it with paraproducts and some commutator lemma as

$$B = \int_{\mathbb{T}_\varepsilon^3} (\mathbb{Y}^2 \succ \Phi_t)\, \mathcal{Q}^{-1} (\mathbb{Y}^2 \succ \Phi_t) = \int_{\mathbb{T}_\varepsilon^3} (\mathbb{Y}^2 \succ \Phi_t) \circ \mathcal{Q}^{-1} (\mathbb{Y}^2 \succ \Phi_t)$$

(only the resonant term counts in integrals)

$$B = \int_{\mathbb{T}_\varepsilon^3} (\mathbb{Y}^2 \circ \mathcal{Q}^{-1} \mathbb{Y}^2)\, \Phi^2 + \underbrace{C'(\mathbb{Y}^2, \mathbb{Y}^2, \Phi, \Phi)}_{\text{nice commutator}}$$

The same considerations as above apply to the explicit polynomial $\mathbb{Y}^2 \circ \mathcal{Q}^{-1} \mathbb{Y}^2$ and one defines

$$\mathbb{Y}^{2 \circ \{2\}} := \mathbb{Y}^2 \circ \mathcal{Q}^{-1} \mathbb{Y}^2 - d_\varepsilon$$

with the same constant as above. It is very similar to $\mathbb{Y}^2 \circ \mathscr{L}^{-1} \mathbb{Y}^2$, in particular the divergent part is the same! (very important). So the analysis of B gives

$$B = \int_{\mathbb{T}_\varepsilon^3} \mathbb{Y}^{2 \circ \{2\}} \Phi^2 + C'(\mathbb{Y}^2, \mathbb{Y}^2, \Phi, \Phi) + \int_{\mathbb{T}_\varepsilon^3} d_\varepsilon \Phi^2 .$$

Putting all together we have

$$\int_{\mathbb{T}_\varepsilon^3} \left[-\frac{9\lambda^2}{4} \int_{\mathbb{T}_\varepsilon^3} (\mathbb{Y}^2 \succ \Phi_t)(m^2 - \Delta_\varepsilon)^{-1} (\mathbb{Y}^2 \succ \Phi_t) - \frac{3}{2} \lambda \Phi (\mathbb{Y}^2 \circ \mathbb{H}) - \frac{1}{2} \beta' \Phi (\mathbb{Y}^1 + Z) \right]$$

$$= -\frac{9\lambda^2}{4} \int_{\mathbb{T}_\varepsilon^3} \mathbb{Y}^{2 \circ \{2\}} \Phi^2 - \frac{9\lambda}{4} C'(\mathbb{Y}^2, \mathbb{Y}^2, \Phi, \Phi)$$

$$-\frac{3\lambda}{2} \int_{\mathbb{T}_\varepsilon^3} \Phi \left[-\frac{\lambda}{2} \mathbb{Y}^{2 \circ [3]} - \frac{3\lambda}{2} \mathbb{Y}^{2 \circ [2]} \mathbb{H} - \frac{3\lambda}{2} C(\mathbb{Y}^2, \mathbb{Y}^2, \mathbb{H}) \right]$$

$$-\frac{9\lambda^2}{4} \int_{\mathbb{T}_\varepsilon^3} \Phi [d_\varepsilon (\mathbb{Y}^1 + \mathbb{H} + \Phi)] - \frac{1}{2} \beta' \Phi (\mathbb{Y}^1 + Z),$$

and now the remarkable fact is that we can choose $\beta' = -9\lambda^2 d_\varepsilon / 2$ in order to cancel the divergences coming from d_ε. This means by choosing appropriately β we can remove all the divergences coming from ill-defined products of irregular Gaussian polynomials.

This is possible because this model is “superrenormalizable”, or also called “subcritical”, i.e. the linear part of the equation dominates the irregular terms in small scales, or said otherwise the non-linear irregular terms can be treated as a perturbation of the linear part.

We are at the point where in our a-priori estimate we do not have any more ugly term, all the products are well defined with the available regularity and the only step remaining is to check that we can close the a-priori estimates, i.e. estimate every term in the l.h.s. with the good terms in the r.h.s.

Let's summarize the discussion of this morning by writing down the final equation which will give rise to our a-priori estimates.

$$
\begin{aligned}
\frac{1}{2}\frac{\partial}{\partial t}\int_{\mathbb{T}_\varepsilon^3}\Phi^2+\int_{\mathbb{T}_\varepsilon^3}[|\nabla_\varepsilon\Psi|^2+m^2\Psi^2]+\frac{\lambda}{2}\int_{\mathbb{T}_\varepsilon^3}\Phi^4 &= \mathcal{A}\\
\mathcal{A} := -\frac{9\lambda^2}{4}\int_{\mathbb{T}_\varepsilon^3}\mathbb{Y}^{2\circ\{2\}}\Phi^2-\frac{9\lambda}{4}C'(\mathbb{Y}^2,\mathbb{Y}^2,\Phi,\Phi)&\\
-\frac{3\lambda}{2}\int_{\mathbb{T}_\varepsilon^3}\Phi\Big[-\frac{\lambda}{2}\mathbb{Y}^{2\circ[3]}-\frac{3\lambda}{2}\mathbb{Y}^{2\circ[2]}\mathbb{H}-\frac{3\lambda}{2}C(\mathbb{Y}^2,\mathbb{Y}^2,\mathbb{H})\Big]&\\
+\lambda D(\Phi,\mathbb{Y}^2,\Phi)-\frac{3}{2}\lambda\int_{\mathbb{T}_\varepsilon^3}\Phi(\mathbb{Y}^2\prec Z+\mathbb{Y}^1Z^2)-\frac{\lambda}{2}\int_{\mathbb{T}_\varepsilon^3}\Phi((\mathbb{H}+\Phi)^3-\Phi^3)&\\
-\underbrace{\Big[\frac{9\lambda^2}{4}+\frac{1}{2}\beta'\Big]}_{=0}\int_{\mathbb{T}_\varepsilon^3}\Phi[d_\varepsilon(\mathbb{Y}^1+\mathbb{H}+\Phi)]&
\end{aligned}
\tag{14}
$$

where we have a series of explicit probabilistic objects

$\mathbb{Y}^1=Y$	$\in\mathscr{C}^{-1/2-\kappa}$	$\mathbb{Y}^{2\circ\{2\}}:=\mathbb{Y}^2\circ\mathcal{Q}^{-1}\mathbb{Y}^2-d_\varepsilon$	$\in\mathscr{C}^{-\kappa}$
$\mathbb{Y}^2=Y^2-c_\varepsilon$	$\in\mathscr{C}^{-1-\kappa}$	$\mathbb{Y}^{2\circ[2]}:=\mathbb{Y}^2\circ\mathscr{L}^{-1}\mathbb{Y}^2-d_\varepsilon$	$\in\mathscr{C}^{-\kappa}$
$\mathbb{Y}^{[3]}=\mathscr{L}^{-1}(Y^3-3c_\varepsilon Y)$	$\in\mathscr{C}^{1/2-\kappa}$	$\mathbb{Y}^{2\circ[3]}:=\mathbb{Y}^2\circ\mathscr{L}^{-1}\mathbb{Y}^3-3d_\varepsilon Y$	$\in\mathscr{C}^{-1/2-\kappa}$
$\mathbb{H}=-\frac{\lambda}{2}\mathbb{Y}^{[3]}-\frac{3\lambda}{2}\mathscr{L}^{-1}(\mathbb{Y}^2\succ\mathbb{H})$,	$\in\mathscr{C}^{1/2-\kappa}$		

(meaning that we can have uniform estimates in the corresponding spaces which do not blow up as $\varepsilon\to 0$). The unknowns $X\in H^{-1/2-\kappa}$, $Z\in H^{1/2-\kappa}$, $\Phi\in H^{1-\kappa}$, $\Psi\in H^1$, satisfying the decomposition

$$
X=\mathbb{Y}^1+\mathbb{H}-\frac{3\lambda}{2}(m^2-\Delta_\varepsilon)^{-1}[\mathbb{Y}^2\succ\Phi]+\Psi \tag{15}
$$

where

$$
Z:=X-\mathbb{Y}^1,
$$

$$
\Phi:=X-\mathbb{Y}^1-\mathbb{H}=\underbrace{-\frac{3\lambda}{2}(m^2-\Delta_\varepsilon)^{-1}[\mathbb{Y}^2\succ\Phi]}_{\in\mathscr{C}^{1-\kappa}=B^{1-\kappa}_{\infty,\infty}}+\underbrace{\Psi}_{\in H^1=B^1_{2,2}}.
$$

With this decomposition one is able to prove that for small $\delta>0$ there exist an explicit function $Q(\mathbb{Y})$ such that

$$
|\mathcal{A}|\leqslant Q(\mathbb{Y})+\delta\Big[\int_{\mathbb{T}_\varepsilon^3}[|\nabla_\varepsilon\Psi|^2+m^2\Psi^2]+\frac{\lambda}{2}\int_{\mathbb{T}_\varepsilon^3}\Phi^4\Big] \tag{16}
$$

(at this point this kind of argument proceed as in $d=2$, i.e. via functional analytic estimates). The function $Q(\mathbb{Y})$ depends only on

$$\mathbb{Y} := (\mathbb{Y}^1, \mathbb{Y}^2, \mathbb{Y}^{[3]}, \mathbb{Y}^{2\circ\{2\}}, \mathbb{Y}^{2\circ[2]}, \mathbb{Y}^{2\circ[3]})$$

via norms of the kind

$$Q(\mathbb{Y}) = Q(\|\mathbb{Y}^1\|_{C([0,T],\mathscr{C}^{-1/2-\kappa})}, \|\mathbb{Y}^2\|_{C([0,T],\mathscr{C}^{-1-\kappa})}, \ldots),$$

in particular

$$\sup_{\varepsilon>0} \mathbb{E}[Q(\mathbb{Y}_\varepsilon)^K] < \infty$$

for any power $K \geqslant 1$.

Using (16) in (14) we get that for δ small enough

Theorem 9. *We have*

$$\frac{1}{2}\frac{\partial}{\partial t}\int_{\mathbb{T}_\varepsilon^3} \Phi^2 + (1-\delta)\left[\int_{\mathbb{T}_\varepsilon^3} [|\nabla_\varepsilon \Psi|^2 + m^2 \Psi^2] + \frac{\lambda}{2}\int_{\mathbb{T}_\varepsilon^3} \Phi^4\right] \leqslant Q(\mathbb{Y}_\varepsilon)$$

provided β is chosen depending on ε in a precise divergent way

$$\beta = C_1 \varepsilon^{-1} + C_2 \lambda \log(\varepsilon^{-1}),$$

with constants C_1, C_2 which we computed above.

As in $d=2$ this can be now used to obtain a-priori estimates for the measure by taking the average and use "stationarity".

Remark. I'm ignoring some technical problem which need to be addressed, in particular one cannot construct stationary solutions to the equation for $\mathbb{H}$, as a consequence both Φ and Ψ are not stationary and the argument to get the appropriate estimates in average has to be modified. But the changes are minor.

Anyway one obtain at the end that for any $t \in [0,T]$,

$$\mathbb{E}\left[\int_{\mathbb{T}_\varepsilon^3} [|\nabla_\varepsilon \Psi_t|^2 + m^2 \Psi_t^2] + \frac{\lambda}{2}\int_{\mathbb{T}_\varepsilon^3} \Phi_t^4\right] \leqslant \mathbb{E}\, Q(\mathbb{Y}_\varepsilon),$$

which given the relation of Ψ, Φ with X allows to obtain tightness, i.e. one can prove that

$$\sup_\varepsilon \mathbb{E}\left[\|X_0^\varepsilon\|_{H^{-1/2}(\mathbb{T}_\varepsilon^3)}^p\right] < \infty,$$

for any $p>1$. And even better, with some more care one can prove that

$$\sup_{\varepsilon} \mathbb{E}[\exp(\beta \|X_0^\varepsilon\|_{H^{-1/2}(\mathbb{T}_\varepsilon^3)}^{1-\kappa})] = \sup_{\varepsilon} \int \exp(\beta \|\varphi\|_{H^{-1/2}(\mathbb{T}_\varepsilon^3)}^{1-\kappa})\, \nu^\varepsilon(\mathrm{d}\varphi) < \infty,$$

for small $\kappa>0$ and $\beta>0$. So the measure ν^ε allows uniform exponential integrability for some power less than 1 of the norm $\|\varphi\|_{H^{-1/2}(\mathbb{T}_\varepsilon^3)}$. This is more than enough to obtain tightness.

6.3 The Φ_3^4 measure without cutoffs

We must now combine the $\varepsilon\to 0$ proof with the $M\to\infty$ proof. This is not difficult but one needs to pay attention to some subtle detail.

Let us start by noting that the above a-priori estimates works also with weights, i.e. instead of testing the elaborated equation with Φ one tests with $\rho^2\Phi$ for some polynomial weight ρ. This make appear weighted Besov norms of the type

$$\|\rho^\sigma \mathbb{Y}^1\|_{C([0,T],\mathscr{C}^{-1/2-\kappa}(\Lambda_{\varepsilon,M}))}, \|\rho^\sigma \mathbb{Y}^2\|_{C([0,T],\mathscr{C}^{-1-\kappa}(\Lambda_{\varepsilon,M}))}, \ldots$$

for some $\sigma>0$, and also norms like

$$\|\rho^{1/2}\Phi\|_{L^4(\Lambda_{\varepsilon,M})}, \quad \|\rho\nabla_\varepsilon\Phi\|_{L^2(\Lambda_{\varepsilon,M})}, \quad \|\rho\,\Phi\|_{L^2(\Lambda_{\varepsilon,M})}$$

for the solution. The first point is to make sure that norms like

$$\|\rho^\sigma \mathbb{Y}^1\|_{C([0,T],\mathscr{C}^{-1/2-\kappa}(\Lambda_{\varepsilon,M}))}$$

are uniformly bounded in M as $M\to\infty$. The idea is that all the processes $(\mathbb{Y}^\tau)_\tau$ growth at infinity at most polynomially with a small power, e.g. one can prove

$$|\Delta_i \mathbb{Y}_t^1(x)| \leqslant C(1+|x|)^\delta(1+|t|)^\delta, \qquad t\in\mathbb{R}, x\in\Lambda_{\varepsilon,M}$$

uniformly in M and ε for some finite random constant C. It is somehow clear that one cannot get better estimates, in particular this kind of stochastic processes cannot be bounded in the full space without weight.

Example. A discrete model. Let $(G_n)_{n\geqslant 1}$ a family of i.i.d $\mathscr{N}(0,1)$, then one can prove that there exists a random constant $C<\infty$ almost surely such that

$$|G_n(\omega)| \leqslant (C(\omega) + c\log^{1/2} n), \qquad n\geqslant 1$$

almost surely for some deterministic constant c. To prove this one shows that

$$Q(\omega) := \sum_{n\geqslant 1} \frac{1}{n^2} e^{\beta|G_n(\omega)|^2}$$

is integrable for small β. This implies that it is finite a.s. and then of course that

$$e^{\beta|G_n(\omega)|^2} \leqslant n^2 Q(\omega), \qquad \Rightarrow \qquad |G_n(\omega)| \leqslant \left(\frac{2}{\beta}\log n + \frac{1}{\beta}\log Q(\omega)\right)^{1/2}$$

for all $n \geqslant 1$.

However the biggest problem come from the equation of $\mathbb{H}$:

$$\mathbb{H} = -\frac{\lambda}{2}\mathbb{Y}^{[3]} - \frac{3\lambda}{2}\mathscr{L}^{-1}(\mathbb{Y}^2 \succ \mathbb{H}),$$

since it cannot be solved in weighted spaces: indeed there is a loss of weight in the estimate of the second term. One can then use a trick to solve this problem. See the paper.

At the end one obtains estimates of the form

$$\sup_{\varepsilon,M} \mathbb{E}^{\varepsilon,M}[\exp(\beta\|\rho X_0^{\varepsilon}\|_{H^{-1/2}(\Lambda_{\varepsilon})}^{1-\kappa})] = \sup_{\varepsilon,M} \int \exp(\beta\|\rho\,\varphi\|_{H^{-1/2}(\Lambda_{\varepsilon})}^{1-\kappa})\, \nu^{\varepsilon,M}(\mathrm{d}\varphi) < \infty. \tag{17}$$

It is well enough to have full tightness both in the $\varepsilon \to 0$ and in the $M \to \infty$ limit, irrespective of the size of λ (can be arbitrary large). This is a fully non-perturbative technique.

In conclusion one obtain accumulation points of the family $(\nu^{\varepsilon,M})$ and then it is easy to prove that any of these acc. points is translation invariant and RP, moreover the estimate (17) allow to prove the technical condition required by the OS reconstruction. Any limit point ν give rise to a translation invariant (no rotation so far), RP and "nice" measure and then to a QFT by OS reconstruction.

We do not have uniqueness, nor rotation invariance. If one can prove uniqueness it should be "easy" to prove rotation invariance.

6.4 Some properties of Φ_3^4

Let's now prove some properties of this measure. First of all that for any $\lambda > 0$ any accumulation point is non-Gaussian.

Remark. Ideally one would like to have non-triviality, i.e. that the corresponding QFT describes interacting particles. A Gaussian measure is trivial.

Let us call ν a arbitrary accumulation point. The first remark is that one can actually prove tightness for the triple of stationary processes $(X_{\varepsilon,M}, Y_{\varepsilon,M}, \mathbb{Y}_{\varepsilon,M}^{[3]})_{\varepsilon,M}$. Let us call

$$(X, Y, \mathbb{Y}^{[3]}),$$

a limit in law of the family. Of course $X_0 \sim \nu$ (i.e. is an accumulation point for $(\nu^{\varepsilon,M})_{\varepsilon,M}$). But we have also the dynamics and a coupling of X, Y, in particular this coupling satisfy the same estimates as we have before the limit, that is

$$\zeta := X - \mathbb{Y}^1 + \frac{\lambda}{2} \mathbb{Y}^{[3]} \in H^{1-\kappa}$$

in particular we have that

$$\left| \Delta_i X - \Delta_i \mathbb{Y}^1 + \frac{\lambda}{2} \Delta_i \mathbb{Y}^{[3]} \right| \approx (2^i)^{-1+\kappa}.$$

This estimate tells us that there exists a coupling between the interacting field X and the free field Y so that X is in "first approximation" given as above. In this very precise sense that if I look at the interacting field $\Delta_i X$ at high-momenta, then I see essentially the free field $\Delta_i Y$ and then a correction $\frac{\lambda}{2} \Delta_i \mathbb{Y}^{[3]}$ coming from the interaction (first-order in perturbation theory) and the something else whose size is well controlled. This is not perturbation theory because λ can be very large.

This is an expression of asymptotic freedom in the UV of the theory.

In order to show that X_0 is not gaussian one can show that the 4-th moment is not given by the usual formula for Gaussians. So we look at four-point function

$$U_4^i(X,X,X,X) := \mathbb{E}[\Delta_i X_0(x)\, \Delta_i X_0(x)\, \Delta_i X_0(x)\, \Delta_i X_0(x)] - 3(\mathbb{E}[\Delta_i X_0(x)\, \Delta_i X_0(x)])^2$$

and we want to prove that

$$U_4(X,X,X,X) \neq 0$$

because this implies that X is non-Gaussian. Note that

$$U_4(Y,Y,Y,Y) = 0$$

since Y is Gaussian. Moreover U_4 is a multilinear function, so we can use our decomposition of X to write

$$X = Y - \frac{\lambda}{2} \mathbb{Y}^{[3]} + \zeta,$$

and

$$U_4(X,X,X,X) = \underbrace{U_4(Y,Y,Y,Y)}_{=0} - 2\,\lambda\, U_4(Y,Y,Y,\mathbb{Y}^{[3]})$$

$$+ c\, U_4(Y,Y,\mathbb{Y}^{[3]},\mathbb{Y}^{[3]}) + c\, U_4(Y,\mathbb{Y}^{[3]},\mathbb{Y}^{[3]},\mathbb{Y}^{[3]}) + c\, U_4(Y,Y,Y,\zeta) + \cdots$$

An explicit computation shows that (both upper and lower bounds)

$$U_4(Y,Y,Y,\mathbb{Y}^{[3]}) \approx \mathbb{E}|\underbrace{\Delta_i Y|^3}_{(2^i)^{3/2}} |\underbrace{\Delta_i \mathbb{Y}^{[3]}}_{(2^i)^{-1/2}}| \approx (2^i)^{3/2-1/2} \approx 2^i,$$

and rough bounds show that

$$|U_4(Y, Y, \mathbb{Y}^{[3]}, \mathbb{Y}^{[3]})| \approx [(2^i)^{1/2+\kappa}]^2[(2^i)^{-1/2+\kappa}]^2 \approx (2^i)^{4\kappa}$$

and all the other terms are as small and cannot compensate for $U_4(Y, Y, Y, \mathbb{Y}^{[3]})$ so finally one deduce that

$$U_4(X, X, X, X) = -2\lambda U_4(Y, Y, Y, \mathbb{Y}^{[3]}) + O((2^i)^{(1/2+5\kappa)}) \neq 0$$

for i large enough and $\lambda \neq 0$. This proves non-Gaussianity of X_0. And actually shows that the correlation functions in the UV are given by first order perturbation theory.

7 Conclusion

The lectures end here. My main goal was to present several topics:

- The basic conceptual structure of QM and link with probability theory via the Euclidean approach
- The meaning of RP as the bridge between QM and EQM or QFT and EQFT.
- How stochastic quantization via a Langevin equation allows to study certain “difficult” measures as push-forward of “easy” Gaussian measures.
- How to control the infinite volume limit via PDE arguments involving weighted spaces.
- How to control the UV limit via paraproducts and decompositions involving paraproducts.
- How divergences can be extracted and matched with local counter-terms in the “bare” interaction.
- How the resulting measure can be analyzed via the decompositions obtained and the coupling with the free theory (despite the fact that Φ_3^4 is not absolutely continuous wrt. the free field).