Lecture 1 | 15.2.2021 | 10:00–12:00 via Zoom

Organisation: Mon, Tue, Thu, Mon, Thu, 10–12, 14–16. Lectures will be recorded and made available to the partecipants.

Web page https://www.iam.uni-bonn.de/abteilung-gubinelli/sq-lectures-milan-ws2021

Plan

- We want to give a meaning to: Quantization (QFT), Euclidean QFT, Stochastic Quantization.
- Quantum mechanics. Observable algebra, states, dynamics, reconstruction theorems in finite dimensions, reflection positivity, Nelson's positivity, examples of reflection positive processes: Gaussian and Markov.
- Euclidean quantum field theory. Feynman–Kac formula. need of infinite volume limit, connection with statistical mechanics and Gibbsian formalism. We will have to deal with probabilistic models of ∞-r.v. We introduce the two basic problems: infinite volume limit and presence of objects which are “irregular” on small scales. We will need to prove the existence of certain limit objects (and eventually some of their properties).
- Stochastic quantization. Langevin dynamics in finite dimensions. Global time existence.
- Control of the infinite volume limit (a typical problem in Statistical Mechanics).
- Control of the UV limit (small scale limit) in finite volume. We need tools to deal with this situation: small scale singularities (products of distributions), renormalization, paracontrolled calculus. Keywords: Products of distributions, paraproducts, Littlewood–Paley decomposition, Schauder estimates (PDE tools). Local existence and uniqueness.
- Put everything together: tightness and existence of the model, integrability and verification of the axioms. IBP formula, meaning of the cube, non-Gaussianity (?)
- If we have time:
  - Weak universality: convergence of microscopic models.
  - Alternative SQ approaches: variational formulation (optimal control), Elliptic SQ, Canonical SQ.

We will use a mix of PDE arguments and probabilistic estimates.

1 Introduction

The aim of these lectures is to illustrate the idea of *stochastic quantization* using as motivating example the construction of the Φ^4_3 Euclidean quantum field theory (EQFT).

(Bosonic) EQFTs are certain (complicated and not very explicit) probability measures μ on spaces of Schwarz distributions $\mathscr{S}'(\mathbb{R}^d)$ over the Euclidean space $\mathbb{R}^d$ which have applications in mathematical physics, in particular in the construction of models of relativistic quantum mechanics, the so called quantum field theories (QFTs).

Definition 1. *A stochastic quantization of a probability measure μ is a pair (F_μ, W) of a map F_μ and a Gaussian r.v. W (on a given probability space) such that the r.v.*

$$\phi = F_\mu(W)$$

has law μ.

The leitmotif of these lectures will be to explore the possibility of obtaining as much information as possible on μ via the stochastic quantization map F_μ.

That W is Gaussian is an (arbitrary) choice, Gaussian variables are very well understood so make life easier for us.

There are various possibilities for the choice of F_μ, some of them quite equivalent in their usefulness. During these lectures we will concentrate on a specific measure μ, the Φ^4_3 Euclidean quantum field theory and on a specific way to construct F_μ first suggested in this context by Parisi and Wu ('84): introducing a fictious time and considering a (over-damped) Langevin dynamics driven by a space–time white noise and with invariant measure μ.

Definition 2. *An EQFT μ is a probability measure on $\mathscr{S}'(\mathbb{R}^d)$ satisfying: reflection positivity, Euclidean invariance and certain moment estimates (to be specified later on).*

For $d=3$ an example of a EQFT in this sense is Φ^4_3. The main theorem we would like to prove in these lectures is

Theorem 3. *There exists a (two-parameter) family $\{\mu\}$ of EQFT on $\mathscr{S}'(\mathbb{R}^3)$. We call them the Φ^4_3 model.*

When $d=2$ there are more examples, but pick $d=3$ because involves some further interesting considerations.

2 Quantum mechanics

QM is a theory of measurement (here). We have observables $\mathscr{O}$ (e.g. experimental setups, gauges, indicators) and a set $\mathscr{S}$ of states of the physical system.

Reference:

- F. Strocchi, *An Introduction to the Mathematical Structure of Quantum Mechanics: A Short Course for Mathematicians*, 2 edition (New Jersey: World Scientific Publishing Company, 2008).
- Lectures: Functional Integration and Quantum Mechanics SS2020: https://www.iam.uni-bonn.de/abteilung-gubinelli/teaching/functional-integration-and-qm-ss20

► To every observable $A \in \mathscr{O}$ and state $\omega \in \mathscr{S}$ we can associate a real number $\omega(A) \in \mathbb{R}$ which is our representation of the measure of A in the state ω.

It is postulated that the empirical mean of repeated results $(x_i)_i$ converges as $N \to \infty$ and the limit is $\omega(A)$:

$$\omega(A) = \lim_{N \to \infty} \frac{1}{N} \sum_{i=1}^{N} x_i.$$

Given two observables A, B I can measure both and sum the results, obtaining $A+B$ and it is natural to assume that $\omega(A+B) = \omega(A) + \omega(B)$, clearly $\mathscr{O}$ is a (real) vector space and states are linear on $\mathscr{O}$. Also if I can measure A then I can also measure any (bounded) function $f(A)$ of A by taking $f(x_i)$ each time I observe x_i.

It is natural to require that $1 \in \mathscr{O}$ and that $\omega(1) = 1$ for all states ω, moreover states are positive, i.e. $\omega(A) \geqslant 0$ if the observable A is positive (e.g. if $A = B^2$ for some B).

I can measure $A^2, B^2, (A+B)^2$ and define a (commutative) product of two observables

$$A * B := (A+B)^2 - A^2 - B^2.$$

The corresponding algebraic structure is called a Jordan algebra. Finite dimensional Jordan algebras can be classified and essentially they are composed of basic blocks given by square self-adjoint matrices with either real, complex, quaternion coefficients, or a finite dimensional Clifford algebra, or the algebra of 3×3 octonionic matrices (the Albert algebra of dimension 27).

► *** Nature seems to have chosen for us ony the obsevable algebras made of complex self-adjoint matrices. (There seems to be no particular mathematical reason for this).

► In the formalization then one take $\mathscr{O}$ to be the algebra of self-adjoint elements of a C^*-algebra $\mathscr{A}$, i.e. a Banach algebra endowed with an involution $a \in \mathscr{A} \mapsto a^* \in \mathscr{A}$ which satisfy the condition $\|a^*a\| = \|a\|^2$ (C^* condition).

A positive element $a \geqslant 0$ is an element of the form $a = b^*b$ for some $b \in \mathscr{A}$. A state ω is a linear functional on $\mathscr{A}$ which is positive on positive elements ($\omega(a^*a) \geqslant 0$) and normalized ($\omega(1) = 1$).

This is true for finite-dimensional complex matrices by taking the involution as Hermitian conjugate and the norm as the operator norm.

Another C^*-algebra: the complex algebra of continuous bounded function on a compact topological space. Think about $C([0,1];\mathbb{C})$ with the sup norm and pointwise multiplication and complex conjugate as adjoint.

► Any commutative C^* algebra is (up to isomorphism) the algebra of bounded continuous functions on a compact space X (Gelfand's theorem).

This implies in particular that for any $x \in X$ we have a state δ_x given by the valution at the point $x \in X$ (the Dirac measure) such that $\delta_x(AB) = \delta_x(A)\delta_x(B) = A(x)B(x)$. Any other state is a convex combination of such states and we have essentially a probability theory. In particular there exists states which corresponds to infinitely precise measurements of all the observables in $\mathscr{O} \subseteq \mathscr{A}$.

► This consequence is at odd with the experience and therefore we have to admit into our modelization non-commutative C^*-algebras.

With non-comm C^* algebra one can make up finite dim examples with two observable such that : if in the state ω one is very precise (small variance) then the other is maximally spread.

► The Gelfand–Naimark theorem states that any C^* algebra can be faithfully represented in Hilbert space $\mathscr{H}$ as a subalgebra of the algebra of all bounded operators $\mathscr{B}(\mathscr{H})$.

Theorem 4. *(GNS) For any state ω there exists a triplet $(\mathscr{H}_\omega, \pi_\omega, \varphi_\omega)$ where $\mathscr{H}_\omega$ is an Hilbert space, π_ω a representation of $\mathscr{A}$ on $\mathscr{H}_\omega$ via bounded operators and a vector $\varphi_\omega \in \mathscr{H}_\omega$ such that, for all $A \in \mathscr{A}$*

$$\omega(A) = \langle \varphi_\omega, \pi_\omega(A)\varphi_\omega \rangle.$$

The representation is irreducible iff the state is pure, i.e. if cannot be written as convex linear combination of other states.

What about dynamics: we are used to discuss dynamics in terms of motions of points, like in classical mechanics. Here we don't have points in general. The right point of view is to introduce an automorphism group $(\alpha_t)_{t\in\mathbb{R}}$ of $\mathscr{A}$, i.e. $\alpha_t(ab) = \alpha_t(a)\alpha_t(b)$, $\alpha_t(a)^* = \alpha_t(a^*)$, $\alpha_{t+s} = \alpha_t \circ \alpha_s$.

The goal of quantum mechanics is therefore to establish which particular triplet $(\mathscr{A}, \omega, \alpha)$ of algebra, state and dynamics allow to predict the behaviour of a given quantum system in a given state. In particular, given the observables $a_1, \ldots, a_n$ and the sequence of times $t_1, \ldots, t_n$ one would like to compute

$$\omega(\alpha_{t_1}(a_1) \cdots \alpha_{t_n}(a_n)).$$

One of the interests of having a specific representation is that it is then possible to proceed to numerical approximations of such quantities.

Symmetries are represented by automorphisms of $\mathscr{A}$, for example time-translations are symmetries.