Lecture 10 | 25.2.2021 | 14:00–16:00 via Zoom

Web page: https://www.iam.uni-bonn.de/abteilung-gubinelli/sq-lectures-milan-ws2021
Recorded lectures: https://uni-bonn.sciebo.de/s/6mTx2gYCfCscfFm

Today:

- Show the second renormalization (not present in $d=2$). [DONE]
- Finish the discussion of apriori estimates, both in finite and then infinite volume, this will give tightness for the measure and existence of accumulation points.
- Give some properties of these accumulations points. (So far we do not have proofs of uniqueness within the SQ approach) Remark: uniqueness is expected when λ/m^2 small enough.

Let's summarize the discussion of this morning by writing down the final equation which will give rise to our apriori estimates.

$$\boxed{\begin{gathered}\frac{1}{2}\frac{\partial}{\partial t}\int_{\mathbb{T}_\varepsilon^3}\Phi^2+\int_{\mathbb{T}_\varepsilon^3}[|\nabla_\varepsilon\Psi|^2+m^2\Psi^2]+\frac{\lambda}{2}\int_{\mathbb{T}_\varepsilon^3}\Phi^4=\mathcal{A}\\ \mathcal{A}:=-\frac{9\lambda^2}{4}\int_{\mathbb{T}_\varepsilon^3}\mathbb{Y}^{2\circ\{2\}}\Phi^2-\frac{9\lambda}{4}C'(\mathbb{Y}^2,\mathbb{Y}^2,\Phi,\Phi)\\ -\frac{3\lambda}{2}\int_{\mathbb{T}_\varepsilon^3}\Phi\left[-\frac{\lambda}{2}\mathbb{Y}^{2\circ[3]}-\frac{3\lambda}{2}\mathbb{Y}^{2\circ[2]}\mathbb{H}-\frac{3\lambda}{2}C(\mathbb{Y}^2,\mathbb{Y}^2,\mathbb{H})\right]\\ +\lambda D(\Phi,\mathbb{Y}^2,\Phi)-\frac{3}{2}\lambda\int_{\mathbb{T}_\varepsilon^3}\Phi(\mathbb{Y}^2\prec Z+\mathbb{Y}^1Z^2)-\frac{\lambda}{2}\int_{\mathbb{T}_\varepsilon^3}\Phi((\mathbb{H}+\Phi)^3-\Phi^3)\\ -\underbrace{\left[\frac{9\lambda^2}{4}+\frac{1}{2}\beta'\right]}_{=0}\int_{\mathbb{T}_\varepsilon^3}\Phi[d_\varepsilon(\mathbb{Y}^1+\mathbb{H}+\Phi)]\end{gathered}} \tag{1}$$

where we have a series of explicit probabilistic objects

$$\begin{array}{llll}\mathbb{Y}^1=Y & \in\mathscr{C}^{-1/2-\kappa} & \mathbb{Y}^{2\circ\{2\}}:=\mathbb{Y}^2\circ\mathscr{Q}^{-1}\mathbb{Y}^2-d_\varepsilon & \in\mathscr{C}^{-\kappa}\\ \mathbb{Y}^2=Y^2-c_\varepsilon & \in\mathscr{C}^{-1-\kappa} & \mathbb{Y}^{2\circ[2]}:=\mathbb{Y}^2\circ\mathscr{L}^{-1}\mathbb{Y}^2-d_\varepsilon & \in\mathscr{C}^{-\kappa}\\ \mathbb{Y}^{[3]}=\mathscr{L}^{-1}(Y^3-3c_\varepsilon Y) & \in\mathscr{C}^{1/2-\kappa} & \mathbb{Y}^{2\circ[3]}:=\mathbb{Y}^2\circ\mathscr{L}^{-1}\mathbb{Y}^3-3d_\varepsilon Y & \in\mathscr{C}^{-1/2-\kappa}\\ \mathbb{H}=-\frac{\lambda}{2}\mathbb{Y}^{[3]}-\frac{3\lambda}{2}\mathscr{L}^{-1}(\mathbb{Y}^2\succ\mathbb{H}), & \in\mathscr{C}^{1/2-\kappa} & & \end{array}$$

(meaning that we can have uniform estimates in the corresponding spaces which do not blow up as $\varepsilon\to 0$). The unknowns $X\in H^{-1/2-\kappa}$, $Z\in H^{1/2-\kappa}$, $\Phi\in H^{1-\kappa}$, $\Psi\in H^1$, satisfying the decomposition

$$X=\mathbb{Y}^1+\mathbb{H}-\frac{3\lambda}{2}(m^2-\Delta_\varepsilon)^{-1}[\mathbb{Y}^2\succ\Phi]+\Psi \tag{2}$$

where

$$Z:=X-\mathbb{Y}^1,$$

$$\Phi:=X-\mathbb{Y}^1-\mathbb{H}=\underbrace{-\frac{3\lambda}{2}(m^2-\Delta_\varepsilon)^{-1}[\mathbb{Y}^2\succ\Phi]}_{\in\mathscr{C}^{1-\kappa}=B^{1-\kappa}_{\infty,\infty}}+\underbrace{\Psi}_{\in H^1=B^1_{2,2}}.$$

With this decomposition one is able to prove that for small $\delta > 0$ there exist an explicit function $Q(\mathbb{Y})$ such that

$$|\mathscr{A}| \leqslant Q(\mathbb{Y}) + \delta\left[\int_{\mathbb{T}_\varepsilon^3} [|\nabla_\varepsilon \Psi|^2 + m^2 \Psi^2] + \frac{\lambda}{2}\int_{\mathbb{T}_\varepsilon^3} \Phi^4\right] \tag{3}$$

(at this point this kind of argument proceed as in $d = 2$, i.e. via functional analytic estimates). The function $Q(\mathbb{Y})$ depends only on

$$\mathbb{Y} := (\mathbb{Y}^1, \mathbb{Y}^2, \mathbb{Y}^{[3]}, \mathbb{Y}^{2\circ\{2\}}, \mathbb{Y}^{2\circ[2]}, \mathbb{Y}^{2\circ[3]})$$

via norms of the kind

$$Q(\mathbb{Y}) = Q(\|\mathbb{Y}^1\|_{C([0,T],\mathscr{C}^{-1/2-\kappa})}, \|\mathbb{Y}^2\|_{C([0,T],\mathscr{C}^{-1-\kappa})}, \ldots),$$

in particular

$$\sup_{\varepsilon > 0} \mathbb{E}[Q(\mathbb{Y}_\varepsilon)^K] < \infty$$

for any power $K \geqslant 1$.

At this point, using (3) in (1) we get that for δ small enough

Theorem 1. *We have*

$$\frac{1}{2}\frac{\partial}{\partial t}\int_{\mathbb{T}_\varepsilon^3} \Phi^2 + (1-\delta)\left[\int_{\mathbb{T}_\varepsilon^3} [|\nabla_\varepsilon \Psi|^2 + m^2 \Psi^2] + \frac{\lambda}{2}\int_{\mathbb{T}_\varepsilon^3} \Phi^4\right] \leqslant Q(\mathbb{Y}_\varepsilon)$$

provided β is chosen depending on ε in a precise divergent way

$$\beta = C_1 \varepsilon^{-1} + C_2 \lambda \log(\varepsilon^{-1}),$$

with constants C_1, C_2 which we computed above.

As in $d = 2$ this can be now used to obtain apriori estimates for the measure by taking the average and use "stationarity".

Remark. I'm ignoring some technical problem which need to be addressed, in particular one cannot construct stationary solutions to the equation for $\mathbb{H}$, as a consequence both Φ and Ψ are not stationary and the argument to get the appropriate estimates in average has to be modified. But the changes are minor.

Anyway one obtain at the end that for any $t \in [0, T]$,

$$\mathbb{E}\left[\int_{\mathbb{T}_\varepsilon^3} [|\nabla_\varepsilon \Psi_t|^2 + m^2 \Psi_t^2] + \frac{\lambda}{2}\int_{\mathbb{T}_\varepsilon^3} \Phi_t^4\right] \leqslant \mathbb{E} Q(\mathbb{Y}_\varepsilon),$$

which given the relation of Ψ, Φ with X allows to obtain tightness, i.e. one can prove that

$$\sup_\varepsilon \mathbb{E}\left[\|X_0^\varepsilon\|_{H^{-1/2}(\mathbb{T}_\varepsilon^3)}^p\right] < \infty,$$

for any $p > 1$. And even better, with some more care one can prove that

$$\sup_\varepsilon \mathbb{E}[\exp(\beta \|X_0^\varepsilon\|_{H^{-1/2}(\mathbb{T}_\varepsilon^3)}^{1-\kappa})] = \sup_\varepsilon \int \exp(\beta \|\varphi\|_{H^{-1/2}(\mathbb{T}_\varepsilon^3)}^{1-\kappa}) \nu^\varepsilon(\mathrm{d}\varphi) < \infty,$$

for small $\kappa > 0$ and $\beta > 0$. So the measure ν^{ε} allows uniform exponential integrability for some power less than 1 of the norm $\|\varphi\|_{H^{-1/2}(\mathbb{T}_{\varepsilon}^3)}$. This is more than enough to obtain tightness.

The Φ_3^4 measure without cutoffs

We must now combine the $\varepsilon \to 0$ proof with the $M \to \infty$ proof. This is not difficult but one needs to pay attention to some subtle detail.

Let us start by noting that the above aprori estimates works also with weights, i.e. instead of testing the elaborated equation with Φ one tests with $\rho^2 \Phi$ for some polynomial weight ρ. This make appear weighted Besov norms of the type

$$\|\rho^{\sigma}\mathbb{Y}^1\|_{C([0,T],\mathscr{C}^{-1/2-\kappa}(\Lambda_{\varepsilon,M}))}, \|\rho^{\sigma}\mathbb{Y}^2\|_{C([0,T],\mathscr{C}^{-1-\kappa}(\Lambda_{\varepsilon,M}))}, \ldots$$

for some $\sigma > 0$, and also norms like

$$\|\rho^{1/2}\Phi\|_{L^4(\Lambda_{\varepsilon,M})}, \quad \|\rho \nabla_{\varepsilon}\Phi\|_{L^2(\Lambda_{\varepsilon,M})}, \quad \|\rho\Phi\|_{L^2(\Lambda_{\varepsilon,M})}$$

for the solution. The first point is to make sure that norms like

$$\|\rho^{\sigma}\mathbb{Y}^1\|_{C([0,T],\mathscr{C}^{-1/2-\kappa}(\Lambda_{\varepsilon,M}))}$$

are uniformly bounded in M as $M \to \infty$. The idea is that all the processes $(\mathbb{Y}^{\tau})_{\tau}$ growth at infinity at most polynomially with a small power, e.g. one can prove

$$|\Delta_i \mathbb{Y}_t^1(x)| \leqslant C(1+|x|)^{\delta}(1+|t|)^{\delta}, \qquad t \in \mathbb{R}, x \in \Lambda_{\varepsilon,M}$$

uniformly in M and ε for some finite random contant C. It is somehow clear that one cannot get better estimates, in particular this kind of stochastic processes cannot be bounded in the full space without weight.

Example. A discrete model. Let $(G_n)_{n\geqslant 1}$ a family of i.i.d $\mathscr{N}(0,1)$, then one can prove that there exists a random constant $C < \infty$ almost surely such that

$$|G_n(\omega)| \leqslant (C(\omega) + c\log^{1/2} n), \qquad n \geqslant 1$$

almost surely for some deterministic constant c. To prove this one shows that

$$Q(\omega) := \sum_{n\geqslant 1} \frac{1}{n^2} e^{\beta|G_n(\omega)|^2}$$

is integrable for small β. This implies that it is finite a.s. and then of course that

$$e^{\beta|G_n(\omega)|^2} \leqslant n^2 Q(\omega), \qquad \Rightarrow \qquad |G_n(\omega)| \leqslant \left(\frac{2}{\beta}\log n + \frac{1}{\beta}\log Q(\omega)\right)^{1/2}$$

for all $n \geqslant 1$.

However the biggest problem come from the equation of $\mathbb{H}$:

$$\mathbb{H} = -\frac{\lambda}{2}\mathbb{Y}^{[3]} - \frac{3\lambda}{2}\mathscr{L}^{-1}(\mathbb{Y}^2 \succ \mathbb{H}),$$

since it cannot be solved in weighted spaces: indeed there is a loss of weight in the estimate of the second term. One can then use a trick to solve this problem. See the paper.

At the end one obtains estimates of the form

$$\boxed{\sup_{\varepsilon,M}\mathbb{E}^{\varepsilon,M}[\exp(\beta\|\rho X_0^{\varepsilon}\|_{H^{-1/2}(\Lambda_\varepsilon)}^{1-\kappa})]=\sup_{\varepsilon,M}\int\exp(\beta\|\rho\varphi\|_{H^{-1/2}(\Lambda_\varepsilon)}^{1-\kappa})\nu^{\varepsilon,M}(\mathrm{d}\varphi)<\infty. \qquad (4)}$$

It is well enough to have full tightness both in the $\varepsilon\to 0$ and in the $M\to\infty$ limit, irrespective of the size of λ (can be arbitrary large). This is a fully non-perturbative technique.

In conclusion one obtain accumulation points of the family $(\nu^{\varepsilon,M})$ and then it is easy to prove that any of these acc. points is translation invariant and RP, moreover the estimate (4) allow to prove the technical condition required by the OS reconstruction. Any limit point ν give rise to a translation invariant (no rotation so far), RP and "nice" measure and then to a QFT by OS recostruction.

We do not have uniqueness, nor rotation invariance. If one can prove uniqueness it should be "easy" to prove rotation invariance.

Some properties of Φ_3^4

Let's now prove some properties of this measure. First of all that for any $\lambda>0$ any accumulation point is non-Gaussian.

Remark. Ideally one would like to have non-triviality, i.e. that the corresponding QFT describes interacting particles. A Gaussian measure is trivial.

Let us call ν a arbitrary accumulation point. The first remark is that one can actually prove tightness for the triple of stationary processes $(X_{\varepsilon,M},Y_{\varepsilon,M},\mathbb{Y}_{\varepsilon,M}^{[3]})_{\varepsilon,M}$. Let us call

$$(X,Y,\mathbb{Y}^{[3]}),$$

a limit in law of the family. Of course $X_0\sim\nu$ (i.e. is an accumulation point for $(\nu^{\varepsilon,M})_{\varepsilon,M}$). But we have also the dynamics and a coupling of X,Y, in particular this coupling satisfy the same estimates as we have before the limit, that is

$$\zeta:=X-\mathbb{Y}^1+\frac{\lambda}{2}\mathbb{Y}^{[3]}\in H^{1-\kappa}$$

in particular we have that

$$\left|\Delta_i X-\Delta_i\mathbb{Y}^1+\frac{\lambda}{2}\Delta_i\mathbb{Y}^{[3]}\right|\approx(2^i)^{-1+\kappa}.$$

This estimate tells us that there exists a coupling between the interacting field X and the free field Y so that X is in "first approximation" given as above. In this very precise sense that if I look at the interacting field $\Delta_i X$ at high-momenta, then I see essentially the free field $\Delta_i Y$ and then a correction $\frac{\lambda}{2}\Delta_i\mathbb{Y}^{[3]}$ coming from the interaction (first-order in perturbation theory) and the something else whose size is well controlled. This is not perturbation theory because λ can be very large.

This is an expression of asymptotic freedom in the UV of the theory.

In order to show that X_0 is not gaussian one can show that the 4-th moment is not given by the usual formula for Gaussians. So we look at four-point function

$$U_4^i(X,X,X,X):=\mathbb{E}[\Delta_i X_0(x)\Delta_i X_0(x)\Delta_i X_0(x)\Delta_i X_0(x)]-3(\mathbb{E}[\Delta_i X_0(x)\Delta_i X_0(x)])^2$$

and we want to prove that

$$U_4(X,X,X,X) \neq 0$$

because this implies that X is non-Gaussian. Note that

$$U_4(Y,Y,Y,Y) = 0$$

since Y is Gaussian. Moverover U_4 is a multilinear function, so we can use our decomposition of X to write

$$X = Y - \frac{\lambda}{2}\mathbb{Y}^{[3]} + \zeta,$$

and

$$U_4(X,X,X,X) = \underbrace{U_4(Y,Y,Y,Y)}_{=0} - 2\lambda U_4(Y,Y,Y,\mathbb{Y}^{[3]})$$

$$+ c\,U_4(Y,Y,\mathbb{Y}^{[3]},\mathbb{Y}^{[3]}) + c\,U_4(Y,\mathbb{Y}^{[3]},\mathbb{Y}^{[3]},\mathbb{Y}^{[3]}) + c\,U_4(Y,Y,Y,\zeta) + \cdots$$

An explicit computation shows that (both upper and lower bounds)

$$U_4(Y,Y,Y,\mathbb{Y}^{[3]}) \approx \mathbb{E}\underbrace{|\Delta_i Y|^3}_{(2^i)^{3/2}}\underbrace{|\Delta_i \mathbb{Y}^{[3]}|}_{(2^i)^{-1/2}} \approx (2^i)^{3/2-1/2} \approx 2^i,$$

and rough bounds show that

$$|U_4(Y,Y,\mathbb{Y}^{[3]},\mathbb{Y}^{[3]})| \approx [(2^i)^{1/2+\kappa}]^2[(2^i)^{-1/2+\kappa}]^2 \approx (2^i)^{4\kappa}$$

and all the other terms are as small and cannot compensate for $U_4(Y,Y,Y,\mathbb{Y}^{[3]})$ so finally one deduce that

$$U_4(X,X,X,X) = -2\lambda U_4(Y,Y,Y,\mathbb{Y}^{[3]}) + O((2^i)^{(1/2+5\kappa)}) \neq 0$$

for i large enough and $\lambda \neq 0$. This proves non-Gaussianity of X_0. And actually shows that the correlation functions in the UV are given by first order perturbation theory.

Wrap up

The lectures end here. My main goal was to present several topics:

- The basic conceptual structure of QM and link with probability theory via the Euclidean approach
- The meaning of RP as the bridge between QM and EQM or QFT and EQFT.
- How stochastic quantisation via a Langevin equation allows to study certain "difficult" mesaures as pushforward of "easy" Gaussian measures.
- How to control the infinite volume limit via PDE arguments involving weighted spaces.
- How to control the UV limit via paraproducts and decompositions involving paraproducts.
- How divergences can be extracted and matched with local counterterms in the "bare" interaction.

- How the resulting measure can be analyzed via the decompositions obtained and the coupling with the free theory (despite the fact that Φ_3^4 is not absolutely continuous wrt. the free field).

These lecture notes are produced using the computer program $\text{T}_{\text{E}}\text{X}_{\text{MACS}}$. If you want to know more go here www.texmacs.org.