Lecture 2 | 15.2.2021 | 14:00–16:00 via Zoom

Web page: https://www.iam.uni-bonn.de/abteilung-gubinelli/sq-lectures-milan-ws2021

Recorded lectures: https://uni-bonn.sciebo.de/s/6mTx2gYCfCscfFm

(we continue the discussion of this morning)

► QM (for us here): a triplet $(\mathcal{A}, \omega, \alpha)$ composed of a C^*-algebra (non-commutative), a state ω and a dynamic one-parameter automorphism group α of $\mathcal{A}$.

Symmetries (i.e. Poincaré invariance) are represented by automorphisms of $\mathcal{A}$, for example time-translations are symmetries.

Given the observables $a_1, \ldots, a_n$ (self-adjoint elements of $\mathcal{A}$) and the sequence of times $t_1, \ldots, t_n$ one would like to compute

$$\omega(\alpha_{t_1}(a_1) \cdots \alpha_{t_n}(a_n)).$$

One of the interests of having a specific representation is that it is then possible to proceed to numerical approximations of such quantities.

Dynamics acts on states by duality $(\alpha_t \omega)(a) = \omega(\alpha_t(a))$.

We will assume that we have an invariant state ω at our disposal, i.e. $\alpha_t \omega = \omega$. We consider the GNS representation $(\mathcal{H}_\omega, \pi_\omega, \varphi_\omega)$. In this case there is a one-parameter group of unitary operators $(U(t))_{t \in \mathbb{R}}$ in $\mathcal{H}_\omega$

$$\pi_\omega(\alpha_t(a)) = U(t)^{-1} \pi_\omega(a) U(t), \qquad U(t) \varphi_\omega = \varphi_\omega,$$

and we assume that they are weakly continuous (and therefore strongly continuous).

In particular we have now for $t_n \leqslant t_{n-1} \leqslant \cdots \leqslant t_1$,

$$\omega(\alpha_{t_1}(a_1) \cdots \alpha_{t_n}(a_n)) = \langle \varphi_\omega, A_1 U(t_1 - t_2) A_2 \cdots U(t_{n-1} - t_n) A_n \varphi_\omega \rangle =: W(\{a_k\}, \{t_k\}) \tag{1}$$

with $A_k = \pi_\omega(a_k)$.

These correlation functions depends only on the differene between times (i.e. they are time-invariant).

One says that φ_ω is cyclic if the span of the vectors of the form

$$A_1 U(t_1 - t_2) A_2 \cdots U(t_{n-1} - t_n) A_n \varphi_\omega$$

for arbitrary As and times, is dense in $\mathcal{H}$.

► Provided the vector φ_ω is cyclic the family of all *correlation functions* of the form (1) for all obsevables and all increasing sets of times form a complete description of the dynamics of a given quantum system, i.e. allow to reconstruct $(\mathcal{H}_\omega, \pi_\omega, \varphi_\omega)$ and also $(U(t))_{t \in \mathbb{R}}$. These are (baby) Wightman functions.

Euclidean quantum mechanics

References

- Barry Simon, *P(ϕ)2 Euclidean (Quantum) Field Theory* (Princeton, N.J: Princeton University Press, 1974).
- James Glimm and Arthur Jaffe, *Quantum Physics: A Functional Integral Point of View*, 2nd ed. (New York: Springer-Verlag, 1987), //www.springer.com/gb/book/9780387964775.

EQM means to do QM in "imaginary time". What does this means precisely?

We need to understand what is the unitary group $U(t)$ at imaginary times.

► By Stone's theorem the strongly continuous unitary group $(U(t))_{t\in\mathbb{R}}$ corresponds to a homomorphism $X: C(\mathbb{R};\mathbb{C}) \to \mathscr{B}(\mathscr{H})$ of C^*-algebra algebras such that $X(e^{it\cdot}) = U(t)$ for all $t \in \mathbb{R}$.

Exercise: check that $(X(e^{it\cdot}))_{t\in\mathbb{R}}$ is a unitary group.

I.e. $f, g \in C(\mathbb{R};\mathbb{C})$ then $X(f), X(g) \in \mathscr{B}(\mathscr{H})$ and $X(f)X(g) = X(fg) = X(g)X(f)$. If $e_t(x) = \exp(itx)$ then $X(e_t) = U(t)$. One could write $f(X) := X(f)$ (its a notation). Then $f(X)g(X) = (fg)(X)$.

I know what does it means to take $t = i\tau$ in $e_t(x)$, that is $x \mapsto e_{i\tau}(x) = \exp(-\tau x)$ but this function is not bounded in general. I would need that $\tau \geqslant 0$ and $x \geqslant 0$.

The group U is of positive energy iff $X(f) = 0$ for all f with support in $\{x < 0 : x \in \mathbb{R}\}$.

► In this case one can define $K(s) = X(e^{-s\cdot}\mathbb{1}_{\mathbb{R}_+}) \in \mathscr{B}(\mathscr{H})$ for all $s \geqslant 0$ (with some care) and observe that $K(t+s) = K(t)K(s)$, $\|K(t)\| \leqslant 1$ and that $t \mapsto K(t)$ is strongly continuous: $(K(t))_{t\geqslant 0}$ is a strongly continuous semigroup of contractions.

► The notable fact is that, given a strongly continuous contractive semigroup $(K(t))_{t\geqslant 0}$ one can reconstructs X and then U so they corresponds each other one-to-one and express essentially the same object, in our case the Hilbert space realisation of the dynamics of the quantum system.

The idea now is to take correlation functions for a dynamics with positive energy and continue it to imaginary time differences to obtain correlation functions of the form

$$S(\{a_k\}, \{t_k\}) := \langle \varphi_\omega, A_1 K(t_1) A_2 \cdots K(t_{n-1}) A_n \varphi_\omega \rangle \tag{2}$$

for arbitrary operators a_k and positive times t_k. We call these Schwinger functions.

Can I go back? $S \to W \to (\mathscr{H}, \pi, \varphi, U)$??

Somewhere inside the family (S) there exists the information of the scalar product in the Hilbert space:

$$S((a_3^*, a_2^*, a_1^*, a_1, a_2, a_3), (t_2, t_1, t_1, t_2)) = \langle \varphi_\omega, A_3^* K(t_2) A_2^* K(t_1) A_1^* A_1 K(t_1) A_2 K(t_2) A_3 \varphi_\omega \rangle$$

$$= \langle A_1 K(t_1) A_2 K(t_2) A_3 \varphi_\omega, A_1 K(t_1) A_2 K(t_2) A_3 \varphi_\omega \rangle \geqslant 0.$$

And many other relations like this. The set of all these positivity properties is called reflection postivity (RP) or Osterwalder–Schräder positivity. The functions S encode (approximations) of all the possible scalar products, and also encode the fact that K is a strongly continuous semigroup of contractions.

The baby version of the Osterwalder–Schräder reconstruction theorem says that:

Theorem. *(OS reconstruction) From the family of functions $\{S(\mathbb{A},\mathbb{T})\}_{\mathbb{A},\mathbb{T}}$ one can recover the QM data, in particular the Wightman functions $\{W(\mathbb{A},\mathbb{T})\}_{\mathbb{A},\mathbb{T}}$ provided these functions satisfy three properties:*

a) Reflection positivity. (encode the geometry of the Hilbert space)

b) Compatibility condition (e.g. encoding the fact that K is a semigroup and other natural algebraic condition)

c) Analytic condition (which encodes the contractivity of the semigroup K + other analytic constraint)

Functions satisfying these properties are called Schwinger functions.

An example of analytic condition is something like:

$$\left|\int S((a_k),(t_1,\ldots,t_n))g_1(t_1)\cdots g_n(t_n)\mathrm{d}t_1\cdots\mathrm{d}t_n\right| \leqslant C_k\left(\prod_k \|a_k\|_{\mathscr{A}}\right)\prod_k\left(\sup_{x\in\mathbb{R}_+}\left|\int e^{-tx}g_k(t)\mathrm{d}t\right|\right). \qquad (3)$$

► So the problem of the construction of interesting QM dynamics is reduced to that of finding Schwinger functions, i.e. correlation functions which satisfy the three above mentioned conditions.

It is a remarkable fact (Nelson, Symanzik,…) that the correlation functions of certain probabilistic models are Schwinger functions.

I want to illustrate this in the simplest case where I have one degrees of freedom, i.e. there exists a particular commutative subalgebra $\mathscr{A}'\subseteq\mathscr{A}$ isomorphic to $C(\mathbb{R};\mathbb{C})$ so I can identify its elements with elements $a\in C(\mathbb{R};\mathbb{C})$. I will use only these obserable algebra to construct Schwinger functions $\{S\}$, and we will assume moreover that

$$S(a_1,\ldots,a_n,t_1,\ldots,t_{n-1}) = \mathbb{E}[a_1(X_{s_1})\cdots a_{n-1}(X_{s_{n-1}})a_n(X_T)]$$

with $s_k = t_k + \cdots + t_{n-1} + T$ for an arbitrary $T\in\mathbb{R}$ where $(X_t)_{t\in\mathbb{R}}$ is a real valued stochastic process (let's say with trajectories in $C(\mathbb{R};\mathbb{R})$). In order for this definition not to depend on T we require that the process X is stationary in time, i.e. the processes $(X_t)_{t\in\mathbb{R}}$ and $(X_{t+s})_{t\in\mathbb{R}}$ have the same law for all $s\in\mathbb{R}$ (we can take $T=0$ in the above definition).

So the question now becomes: under which conditions on the law of X these Schwinger functions have the form (2) for some quantum data and contractive semigroup K?

It is not difficult to show that the family of such functions satisfy the compatibility conditions in Theorem 2.

Reflection positivity is trickier. We need to formulate it in the probabilistic language.

On functions F on on $C(\mathbb{R};\mathbb{R})$ we can introduce an operation Θ of time inversion such that $(\Theta F)(x) = F(\theta x)$ with $(\theta x)(t) = x(-t)$ is the time-reflection of the path $x\in C(\mathbb{R};\mathbb{R})$.

Then for any complex function F on $C(\mathbb{R}_{\geqslant 0};\mathbb{R})$ of the form

$$F(x) = \sum_k c_k e^{i\lambda_k x(t_k)} \qquad x\in C(\mathbb{R}_{\geqslant 0};\mathbb{R}) \qquad (4)$$

with coefficients $c_k \in \mathbb{C}$, $\lambda_k \in \mathbb{R}, t_k \in \mathbb{R}_{\geqslant 0}$ (note that the times are positive!), the RP of the Schwinger functions become the relation

$$\mathbb{E}[\overline{\Theta F(X)}\, F(X)] \geqslant 0. \tag{5}$$

Definition 1. *A measure on $C(\mathbb{R};\mathbb{R})$ is reflection positive iff eq. (5) holds for any cylinder function supported on positive times.*

This is already a nontrivial condition.

Let's assume X is Gaussian and stationary. Then it has to be centered and with covariance $C(t) = \mathbb{E}[X_t X_0]$ such that

$$0 \leqslant \mathbb{E}\left[\left(\sum_k \bar{c}_k X_{-t_k}\right)\left(\sum_k c_k X_{t_k}\right)\right] = \sum_{k,k'} c_k \bar{c}_{k'} C(t_k + t_{k'})$$

for any $c_k \in \mathbb{C}$ and $t_k \geqslant 0$. Functions satisfying this property are called totally monotone.

By a theorem of Bernstein a bounded and totally monotone function $C: \mathbb{R}_{\geqslant 0} \to \mathbb{C}$ has the representation

$$C(t) = \int_{\mathbb{R}_{\geqslant 0}} e^{-tx} \nu(\mathrm{d}x)$$

for some bounded positive measure ν on $\mathbb{R}_+$. In this case, indeed,

$$\sum_{k,k'} c_k \bar{c}_{k'} C(t_k + t_{k'}) = \int_{\mathbb{R}_{\geqslant 0}} \left|\sum_k c_k e^{-t_k x}\right|^2 \nu(\mathrm{d}x) \geqslant 0.$$

The simplest case is when the measure ν is concentrated in a point $\alpha > 0$, then we have the covariance

$$C(t) = \frac{e^{-\alpha|t|}}{2\alpha}, \qquad t \in \mathbb{R},$$

with an arbitrary normalization (see below).

The corresponding Gaussian process is called the Ornstein–Uhlenbeck process (OU) and we conclude that any scalar reflection positive Gaussian process in one dimension can be constructed by taking sums of independent OU processes (see the QMFI lecture notes).

Symmetric Markov processes

Another important strategy to obtain RP processes is to use Markovianity. Let X be a Markov process, i.e. such that

$$\mathbb{E}[f(X_t)|\mathscr{F}_s] = \mathbb{E}[f(X_t)|X_s],$$

where $(\mathscr{F}_s = \sigma(X_r : r \leqslant s))_{s \in \mathbb{R}}$ is the filtration generated by X. Recalling the support of the function F in the RP condition, we have

$$\mathbb{E}[\overline{\Theta F} F] = \mathbb{E}[\mathbb{E}[\overline{\Theta F} F|\mathscr{F}_0]] = \mathbb{E}[\overline{\Theta F}\, \mathbb{E}[F|\mathscr{F}_0]] \underset{\text{Markovianity}}{=\!=\!=} \mathbb{E}[\overline{\Theta F}\, \mathbb{E}[F|X_0]]$$

$$= \mathbb{E}[\overline{\mathbb{E}[\Theta F|X_0]}\, \mathbb{E}[F|X_0]]$$

An easy condition to force positivity of this quantity is to assume that the law of X is symmetric wrt. time inversion, i.e. $X \sim \theta X$, in this case $\mathbb{E}[\Theta F|X_0] = \mathbb{E}[F|X_0]$ and we have

$$\mathbb{E}[\overline{\Theta F(X)}\, F] = \mathbb{E}[\overline{\mathbb{E}[F(X)|X_0]}\, \mathbb{E}[F(X)|X_0]] = \mathbb{E}[|\,\mathbb{E}[F(X)|X_0]|^2] \geqslant 0.$$

Indeed

$$\mathbb{E}[\mathbb{E}[\Theta F(X)|X_0] g(X_0)] = \mathbb{E}[F(\theta X) g(X_0)] = \mathbb{E}[F(\theta X) g(\theta X_0)] \underset{\text{symmetry}}{=\!=\!=} \mathbb{E}[F(X) g(X_0)]$$

$$= \mathbb{E}[\mathbb{E}[F(X)|\mathscr{F}_0] g(X_0)] \underset{\text{Markov.}}{=\!=\!=} \mathbb{E}[\mathbb{E}[F(X)|X_0] g(X_0)]$$

for arbitrary g so indeed $\mathbb{E}[\Theta F(X)|X_0] = \mathbb{E}[F(X)|X_0]$.

► Stationary and time-reversal invariant Markov processes are reflection positive. The converse is also true: RP Markov processes are time-reversal invariant and stationary.