Lecture 3 | 16.2.2021 | 10:00–12:00 via Zoom

Web page: https://www.iam.uni-bonn.de/abteilung-gubinelli/sq-lectures-milan-ws2021
Recorded lectures: https://uni-bonn.sciebo.de/s/6mTx2gYCfCscfFm

Recall from yesterday:

Definition 1. *A real valued continuous stochastic process* $(X_t)_{t\in\mathbb{R}}$ *is RP iff for any bounded function* $F: C(\mathbb{R}_{\geqslant 0};\mathbb{R})\to\mathbb{C}$ *one has*

$$\mathbb{E}[\overline{F(\theta X)}F(X)]\geqslant 0,$$

where $(\theta X)_t = X_{-t}$.

And we saw that among Gaussian processes only linear combitations of OU processes satisfy this condition and morever that symmetric Markov processes satisfy this condition.

We are going to give a sketch of the reconstruction of the QM data from a symmetric Markov process.

We consider the complex vector space $\mathscr{E}_+$ of cylindrical functions supported on positive times, i.e. functions of the form

$$F=\sum_k c_k e^{i\lambda_k X(t_k)},$$

for $c_k\in\mathbb{C}, \lambda_k\in\mathbb{R}, t_k\geqslant 0$.

► RP endows this space with a Hermitian scalar product

$$\langle F,G\rangle = \mathbb{E}[\overline{\Theta F}G], \qquad F,G\in\mathscr{E}_+,$$

with $\Theta F = F\circ\theta$. We take the quotient $\mathscr{E}_+\setminus\mathscr{N}$ where $\mathscr{N}$ is the subspace of elements in $\mathscr{E}_+$ with zero norm and complete wrt. the scalar product to obtain an Hilbert space $\mathscr{H}$.

► Our algebra of observables $C(\mathbb{R},\mathbb{C})$ acts as multiplication operators i.e. $Q(a)F = a(X_0)F$.

Important observation: the elemens of the from $F-\mathbb{E}[F|\mathscr{F}_0]$ have zero norm.

(Exercise using the Markov property).

In particular for cylinder function

$$F=\sum_k c_k e^{i\lambda_k X(t_k)} \approx \mathbb{E}[F|X_0] = \sum_k c_k \mathbb{E}[e^{i\lambda_k X(t_k)}|X_0] = \sum_k c_k P_{t_k}(e^{i\lambda_k \cdot})(X_0)$$

where P_t is the transition operator of the Markov process $(P_t f)(x) = \mathbb{E}[f(X_t)|X_0 = x]$.

Moreover we have the natural action of time-translation via $t\geqslant 0$: T_t which acts on $\mathscr{E}_+$ (since for example $T_t X_s = X_{s+t}$ with $s+t\geqslant 0$ if $s,t\geqslant 0$).

T_t is a symmetric operator

$$\langle F, T_t G\rangle = \mathbb{E}[\overline{\Theta F}\,T_t G] = \mathbb{E}[T_t((T_{-t}\overline{\Theta F})\,G)] = \mathbb{E}[(T_{-t}\overline{\Theta F})\,G] = \mathbb{E}[(\overline{\Theta T_t F})\,G] = \langle T_t F, G\rangle$$

by stationarity of the law and the definition of Θ. This also implies that $T_t\mathscr{N}\subseteq\mathscr{N}$. We want to show that T_t is contractive. Indeed

$$\langle T_t F, T_t F\rangle = \langle F, T_{2t}F\rangle \leqslant \|F\|_{\mathscr{H}}^{1/2}\|T_{2t}F\|_{\mathscr{H}}^{1/2}$$

and iterating this we have

$$\|T_t F\|_{\mathscr{H}}^2 \leqslant \|F\|_{\mathscr{H}}^{1/2} \|T_{2t} F\|_{\mathscr{H}}^{1/2} \leqslant \cdots \leqslant \|F\|^{1/2+\cdots+1/2^n} \|T_{2^n t} F\|^{1/2^n} \to \|F\|$$

since

$$\|T_{2^n t} F\|^2 = \mathbb{E}[T_{-2^n t} \overline{\Theta F}\, T_{2^n t} F] \leqslant (\mathbb{E}[|T_{-2^n t} \overline{\Theta F}|^2])^{1/2} (\mathbb{E}[|T_{2^n t} F|^2])^{1/2}$$

$$= (\mathbb{E}[T_{-2^n t} |\overline{\Theta F}|^2])^{1/2} (\mathbb{E}[T_{2^n t} |F|^2])^{1/2} = (\mathbb{E}[|\overline{\Theta F}|^2])^{1/2} (\mathbb{E}[|F|^2])^{1/2} < \infty$$

uniformly in t. So we have proved that $\|T_t\| \leqslant 1$.

Strong continuity of $(T_t)_{t \geqslant 0}$ follows then via approximations from weak continuity and from the fact that the the process X is continuous in distribution (Exercise).

The ground state $\varphi = 1 \in \mathscr{H}$, indeed $T_t 1 = 1$ and this also show that $\|T_t\| = 1$.

So we have constructed an Hilbert space $\mathscr{H}$, an $*$-homeomorphism $Q: C(\mathbb{R}, \mathbb{C}) \to \mathscr{B}(\mathscr{H})$ and a strongly continuous contraction semigroup $(T_t)_{t \geqslant 0}$ (from which we can recover a positive energy unitary group $(U(t))_{t \geqslant 0}$ and another *-homeomorphism $H: C(\mathbb{R}_{\geqslant 0}, \mathbb{C}) \to \mathscr{B}(\mathscr{H})$ such that $H(e^{-it\cdot}) = U(t)$ for $t \in \mathbb{R}$ and $H(e^{-t\cdot}) = T_t$ for $t \geqslant 0$.

What is this Hilbert space $\mathscr{H}$ concretely? By what we saw above we know that we can replace any vector with its projection onto $\sigma(X_0)$, that is any vector is just a function of X_0:

$$\langle F, G \rangle_{\mathscr{H}} = \mathbb{E}[\overline{\Theta F}\, G] = \mathbb{E}[\overline{\mathbb{E}[F | X_0]}\, \mathbb{E}[G | X_0]]$$

$$= \mathbb{E}[\overline{f(X_0)}\, g(X_0)] = \int_{\mathbb{R}} \overline{f(x)}\, g(x) \rho(\mathrm{d}x) = \langle f, g \rangle_{L^2(\mathbb{R}, \rho)}$$

where ρ is the law of X_0 and f, g are functions so that $f(X_0) = \mathbb{E}[F | X_0]$, $g(X_0) = \mathbb{E}[G | X_0]$.

So $\mathscr{H} \approx L^2(\mathbb{R}, \rho)$ and $Q(a) f(x) = a(x) f(x)$, but the action of T_t is more complicated in this representation, indeed $(T_t f) = f(X_t)$ and then $T_t f \approx \mathbb{E}[f(X_t) | X_0] = (P_t f)(X_0)$ so we have that it acts on $L^2(\mathbb{R}, \rho)$ as a semigroup $(K(t))_{t \geqslant 0}$

$$(K(t) f)(x) = (P_t f)(x), \qquad x \in \mathbb{R},$$

indeed

$$\langle F, T_t G \rangle_{\mathscr{H}} = \langle f, K(t) g \rangle_{L^2(\mathbb{R}, \rho)}.$$

An explicit example

Let's take the OU process, i.e. the centred Gaussian process $(X_t)_{t \in \mathbb{R}}$ with covariance

$$C(t) = \frac{e^{-\alpha |t|}}{2\alpha}$$

with $\alpha > 0$. This is a symmetric Markov process, it also solves the SDE

$$\mathrm{d}X_t = -\alpha X_t \mathrm{d}t + \mathrm{d}B_t, \qquad X_0 \sim \rho := \mathscr{N}(0, (2\alpha)^{-1})$$

where B is standard Brownian motion. We can compute its transition function

$$(K(t) f)(x) = \mathbb{E}[f(X_t) | X_0 = x] = \mathbb{E}\left[f\left(e^{-\alpha t} x + \left(\frac{1 - e^{-2\alpha t}}{2\alpha}\right)^{1/2} Z\right)\right], \qquad x \in \mathbb{R}, t \geqslant 0,$$

where $Z \sim \mathcal{N}(0,1)$. Note that $K(t)f \to \int f \mathrm{d}\rho$ as $t \to \infty$.

In this example I can perform explicitly the construction of U and Q. We can diagonalize K: Take $e_\lambda(x) = \exp(i\lambda x)$ then

$$(K(t)e_\lambda)(x) = \mathbb{E}\left[e_\lambda\left(e^{-\alpha t}x + \left(\frac{1-e^{-2\alpha t}}{2\alpha}\right)^{1/2} Z\right)\right] = e_\lambda(e^{-\alpha t}x)\exp\left(-\frac{\lambda^2}{2}\left(\frac{1-e^{-2\alpha t}}{2\alpha}\right)\right)$$

so if we let $\hat{e}_\lambda(x) = \exp(i\lambda x)\exp\left(\frac{\lambda^2}{2}\frac{1}{2\alpha}\right)$ we have

$$K(t)\hat{e}_\lambda = \hat{e}_{\lambda e^{-\alpha t}}.$$

By expanding both the l.h.s. and r.h.s in powers of λ we obtain

$$\sum_{n\geqslant 0} \frac{(i\lambda)^n}{n!} K(t)H_n(x) = (K(t)\hat{e}_\lambda)(x) = \hat{e}_{\lambda e^{-\alpha t}}(x) = \sum_{n\geqslant 0} \frac{(i\lambda)^n}{n!} e^{-\alpha n t} H_n(x)$$

where we denoted $H_n(x)$ the coefficients in this expansion (we leave implicit the dependence on α, $H_n(x;\alpha)$). So we learn that

$$K(t)H_n(x) = e^{-\alpha n t}H_n(x)$$

i.e. the Hermite polynomials H_n are eigenfunctions of $K(t)$. They are the coefficients in the series of

$$\exp(i\lambda x)\exp\left(\frac{\lambda^2}{2}\frac{1}{2\alpha}\right) = \sum_{n\geqslant 0} \frac{(i\lambda)^n}{n!} H_n(x).$$

Since $K(t)$ is symmetric then the H_n are orthogonal and actually a basis for $L^2(\rho)$ since if $f \in L^2(\rho)$ such that $0 = \langle f, H_n \rangle$ for all n this implies $\langle f, e_\lambda \rangle = 0$ for all $\lambda \in \mathbb{R}$ (essentially by density and approximation).

At this point we construct easily both E and U, indeed

$$E(a)H_n(x) = a(\alpha n)H_n, \qquad U(t)H_n = e^{i\alpha n t}H_n.$$

► We have completely solved the QM problem for this model: we have an Hilbert space $L^2(\rho)$, a representation Q of $C(\mathbb{R})$ given by multiplication operators, a strongly continuous unitary dynamical group $(U(t))_{t\in\mathbb{R}}$ and a ground state given by the constant function 1.

Now one see that $E(a)$ only cares about the value of the function $a \in C(\mathbb{R}_{\geqslant 0}; \mathbb{C})$ on the set $\alpha\mathbb{N}$, so it is actually an homeomorphism from $C(\alpha\mathbb{N}; \mathbb{C})$. The quantity E is quantized : can only take discrete values, moreover $E(a)$ is constant in time: $U(t)^{-1}E(a)U(t) = E(a)$. It is a constant of motion (of course it is the “energy”).

Exercise: Show that $Q(a)$ and $E(b)$ do not commute for any $a, b \in C(\mathbb{R}; \mathbb{C})$.

How we use all this to solve the original problem: i.e. compute results of measurements. Let ω the state corresponding to the vector $1 \in \mathscr{H} \approx L^2(\rho)$.

$$\omega(\alpha_t(a) * a) = 2\mathrm{Re}\langle 1, U(t)^{-1}Q(a)\, U(t)Q(a)\, 1\rangle = 2\mathrm{Re}\langle 1, Q(a)\, U(t)Q(a)\, 1\rangle$$

$$= 2\mathrm{Re}\sum_n e^{i\alpha t n}\langle 1, Q(a)\, H_n\rangle\langle H_n, Q(a)1\rangle = 2\mathrm{Re}\sum_n e^{i\alpha t n}|\langle H_n, Q(a)1\rangle|^2.$$

So in this system one observes only frequencies of the form $\alpha\mathbb{N}$.

Perturbations of RP processes

As we have said, it is useful to have a source of RP processes which do not rely on Markovianity and survive the generalisation to infinite dimensions which we are going to pursue later on.

A convenient way to construct a large class of RP processes is to take Gibbsian perturbations of a RP process $\mathbb{P}$ ($\mathbb{P}$ is the law of an RP process with $X(\omega)=\omega$ the canonical process).

With this we mean consider a potential function $V:\mathbb{R}\to\mathbb{R}_{\geqslant 0}$ and a new probability measure $\mathbb{Q}$ given by

$$\mathbb{Q}_T(\mathrm{d}\omega)=\frac{1}{Z_T}\exp\left(-\int_{-T}^{T}V(X_s(\omega))\mathrm{d}s\right)\mathbb{P}(\mathrm{d}\omega)$$

with a normalization factor Z_T.

Lemma 2. *The measure $\mathbb{Q}_T$ is reflection positive.*

Proof. Take

$$G=\exp\left(-\int_0^T V(X_s)\mathrm{d}s\right),$$

then

$$\Theta G=\exp\left(-\int_0^T V(X_{-s})\mathrm{d}s\right)=\exp\left(-\int_{-T}^{0} V(X_s)\mathrm{d}s\right)$$

and

$$\exp\left(-\int_{-T}^{T}V(X_s(\omega))\mathrm{d}s\right)=(\Theta G)G=(\overline{\Theta G})G$$

then for any $F\in\mathscr{E}_+$ we have

$$\mathbb{E}_{\mathbb{Q}_T}[\overline{\Theta F}F]=\frac{1}{Z_T}\mathbb{E}_{\mathbb{P}}\left[\overline{\Theta F}F\exp\left(-\int_{-T}^{T}V(X_s(\omega))\mathrm{d}s\right)\right]$$

$$=\frac{1}{Z_T}\mathbb{E}_{\mathbb{P}}[\overline{\Theta F}F\,(\overline{\Theta G})G]=\frac{1}{Z_T}\mathbb{E}_{\mathbb{P}}[\overline{\Theta(FG)}(FG)\,]\geqslant 0$$

since $\mathbb{P}$ is RP. □

But now the problem is that $\mathbb{Q}_T$ is not stationary (i.e. invariant under translation).

Remark. More “fancy” perturbations, like e.g.

$$\exp\left(-\int_{-T}^{T}\int_{-T}^{T}W(X_s(\omega),X_{s'}(\omega))\mathrm{d}s\mathrm{d}s'\right)$$

are not in general reflection positive. Note also that to prove RP we used essentially that there is an integral over time and the multiplicativity of the exponential.

These lecture notes are produced using the computer program $\mathrm{T_EX_{MACS}}$. If you want to know more go here `www.texmacs.org`.