Lecture 4 | 16.2.2021 | 14:00–16:00 via Zoom

Web page: https://www.iam.uni-bonn.de/abteilung-gubinelli/sq-lectures-milan-ws2021
Recorded lectures: https://uni-bonn.sciebo.de/s/6mTx2gYCfCscfFm

Perturbations of RP processes (continuation)

Let $\mathbb{P}$ be the law of RP process on the path space $C(\mathbb{R};\mathbb{R})$ and $X(\omega)=\omega$, with this we mean consider a potential function $V:\mathbb{R}\to\mathbb{R}_{\geqslant 0}$ and a new probability measure $\mathbb{Q}$ given by

$$\mathbb{Q}_T(\mathrm{d}\omega)=\frac{1}{Z_T}\exp\left(-\int_{-T}^{T}V(X_s(\omega))\mathrm{d}s\right)\mathbb{P}(\mathrm{d}\omega),$$

with a normalization factor Z_T.

Lemma. *The measure $\mathbb{Q}_T$ is reflection positive.*

However $\mathbb{Q}_T$ is not translation invariant anymore. In order to adjust this problem we could try to take $T\to\infty$.

Indeed

$$\mathbb{E}_{\mathbb{Q}_T}[T_tF]=\mathbb{E}_{\mathbb{P}}\left[(T_tF)\frac{1}{Z_T}\exp\left(-\int_{-T}^{T}V(X_s)\mathrm{d}s\right)\right]=\mathbb{E}_{\mathbb{P}}\left[F\frac{1}{Z_T}\exp\left(-\int_{-T}^{T}V(X_{s-t})\mathrm{d}s\right)\right]$$
$$=\mathbb{E}_{\mathbb{P}}\left[F\frac{1}{Z_T}\exp\left(-\int_{-T-t}^{T-t}V(X_s)\mathrm{d}s\right)\right]$$

so we can hope to have something similar to the original only if $T\gg 1$. As $T\to\infty$ there is not opportunity for the density to have well defined meaning, so actually we can only hope that the family $(\mathbb{Q}_T)_T$ coverges weakly as $T\to\infty$ (i.e. when testing with smooth functions which depends only at the local behaviour of X in a compact neighborhood of the origin, this defines the correct topology for this limit) because in that case we can hope that if $\mathbb{Q}$ is the unique limit then

$$\mathbb{E}_{\mathbb{Q}}[T_tF]=\mathbb{E}_{\mathbb{Q}}[F].$$

It is clear that any accomulation point will be RP. Indeed the point of working with RP is that it easily pass to the limit in this kind of arguments (provided you have it at the approximation level in the first place).

Can we expect to have a limit? How does it looks like?

Let us make connection with the Markovian point of view. Let assume that $\mathbb{P}$ is a Markov process with semigroup $(K(t))_{t\geqslant 0}$. Introduce a new semigroup $(Q^V(t))_{t\geqslant 0}$ (depending on the "potential" V) as: for any $f\in C(\mathbb{R};\mathbb{C})$ we define

$$(Q^V(t)f)(x)=\mathbb{E}_{\mathbb{P}}\left[f(X_t)\exp\left(-\int_0^t V(X_s)\mathrm{d}s\right)\middle|X_0=x\right].$$

(Feynman–Kac–Nelson formula) Exercise: prove that it is a semigroup (with the Markov property). It is positive $f\geqslant 0\Rightarrow Q^Vf\geqslant 0$ and symmetric (as an operator in the complex Hilbert space $L^2(\theta)$ with $\theta=\mathrm{Law}_{\mathbb{P}}(X_0)$)

$$\langle g,Q_t^Vf\rangle=\mathbb{E}_{\mathbb{P}}\left[\overline{g(X_0)}\exp\left(-\int_0^t V(X_s)\mathrm{d}s\right)f(X_t)\right]=\mathbb{E}_{\mathbb{P}}\left[\overline{g(X_{-t})}\exp\left(-\int_{-t}^0 V(X_s)\mathrm{d}s\right)f(X_0)\right]=\langle Q_t^Vg,f\rangle$$

using stationarity and time-symmetry of $\mathbb{P}$.

Assume that $\|Q_s^V\| < \infty$ for some $s > 0$ (this is true, e.g. if V is bounded below). Then

$$|\langle g, Q_{2s}^V f\rangle| = |\langle Q_s^V g, Q_s^V f\rangle| \leqslant \|Q_s^V\|^2 \|f\| \|g\| \Rightarrow \|Q_{2s}^V\| \leqslant \|Q_s^V\|^2$$

conversely

$$|\langle Q_{s/2}^V g, Q_{s/2}^V f\rangle| = |\langle g, Q_s^V f\rangle| \leqslant \|Q_s^V\| \|f\| \|g\| \Rightarrow \|Q_{s/2}^V\|^2 \leqslant \|Q_s^V\|$$

so iterating these inequalities we have $\|Q_{2^n s}^V\| \leqslant \|Q_s^V\|^{2^n}$, and $\|Q_{s2^{-n}}^V\| \leqslant \|Q_s^V\|^{2^{-n}}$.

For any $t \geqslant 0$ we can write down the dyadic decomposition of $p = t/s$ and get

$$\|Q_t^V\| \leqslant \|Q_s^V\|^p \leqslant \|Q_s^V\|^{t/s}.$$

We have proven that there exists a constant

$$c = \frac{1}{s} \log \|Q_s^V\|$$

such that $\|Q_t^V\| \leqslant e^{ct}$. By repeating the argument with t we have now $\|Q_s^V\|^{1/s} \leqslant \|Q_t^V\|^{1/t} \leqslant \|Q_s^V\|^{1/s}$ so we must have $\|Q_t^V\| = e^{ct}$.

We can then define $K^V(t) := e^{-ct} Q_t^V$ and obtain a contractive semigroup $K^V(t)$ for which $\|K^V(t)\| = 1$ for all $t \geqslant 0$. In particular there must exists at least one vector ψ such that $K^V(t)\psi = \psi$ and it can be chosen to be positive $\psi \geqslant 0$.

Let me remark that, this allow to prove that

$$\frac{1}{2T} \log Z_T = \frac{1}{2T} \log \mathbb{E}\left[\exp\left(-\int_{-T}^{T} V(X_s) \mathrm{d}s\right)\right] = \frac{1}{2T} \log \langle 1, Q_{2T}^V 1\rangle = \frac{1}{2T} \log \langle 1, e^{c2T} K_{2T}^V 1\rangle \to c.$$

Wrapping up this part: in the Markovian setting we can take a potential V and any RP Markov process with semigroup K and construct another stationary symmetric Markov process with semigroup $(K^V(t))_{t \geqslant 0}$ and invariant state ψ in the Hilbert space $\mathcal{H} = L^2(\theta)$.

This gives rise to a new QM model according to the by now familiar procedure giving a new unitary dynamics $(U^V(t))_{t \in \mathbb{R}}$.

If you want to know more (for QM) you can look at, e.g.

József Lörinczi, Fumio Hiroshima, and Volker Betz, *Feynman-Kac-Type Theorems and Gibbs Measures on Path Space: With Applications to Rigorous Quantum Field Theory*, De Gruyter Studies in Mathematics 34 (Berlin; Boston: De Gruyter, 2011).

Euclidean Quantum Field Theory

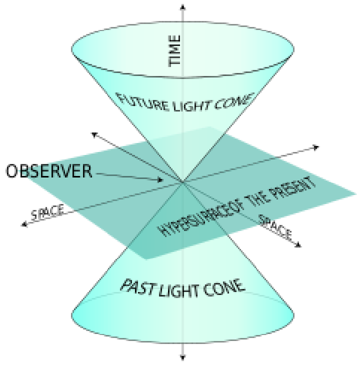
Some reference for this section are:

- R. Haag, *Local Quantum Physics: Fields, Particles, Algebras*, 2nd rev. and enl. ed, Texts and Monographs in Physics (Springer, 1996), http://gen.lib.rus.ec/book/index.php?md5=E91268014A4AF250E095387BD5C2A678.

- F. Strocchi, *An Introduction to Non-Perturbative Foundations of Quantum Field Theory* (Oxford: OUP Oxford, 2013).

How we extend all these considerations to QFTs instead of just QM models? This will involve to consider QM with infinitely many degrees of freedom. (Recall that so far we gave only explicit examples with one degree of freedom, i.e. we were measuing only one quantity at any given time $\mathrm{Hom}(C(\mathbb{R},\mathbb{C}),\mathscr{A})$).

In $(n+1)$-dimensional Minkowski space $\mathbb{M}^{n+1}$ the situation looks like this:



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So we need to be more specific on *where* we measure things, since measurements separated by space-like vectors should not interfere with each other.

More abstractly we could associate to any compactly supported smooth test function $f \in \mathscr{D}(\mathbb{R}^n)$ in n-dimensional space a one-dimensional obsevable algebra (for example), i.e. an $*$-homeomorphism $\Phi_0(f): C(\mathbb{R};\mathbb{C}) \to \mathscr{A}$ into the C^*-algebra $\mathscr{A}$ generated by all observables.

By writing the more suggestive notation $F(\Phi_0(f)) := \Phi_0(f)(F)$ for any $F \in C(\mathbb{R};\mathbb{C})$ we would like to impose that

$$[F(\Phi_0(f)), G(\Phi_0(g))] = 0$$

for each $f, g \in \mathscr{D}(\mathbb{R}^n)$ with disjoint support and $F, G \in C(\mathbb{R};\mathbb{C})$.

All together we then expect a representation of the full Poincaré group by automorphisms of $\mathscr{A}$.

For example if we assume that we can make measurements at a precise point, we can assume to have observables $F(\Phi(t,x))$ with measures a given quantity at time $t \in \mathbb{R}$ and in the point $x \in \mathbb{R}^n$, $(t,x) \in \mathbb{M}^n$. The an element g of the Poincaré group (i.e. the symmetry group of $\mathbb{M}^n$) acts via an automorshism α_g such that

$$\alpha_g(F(\Phi(t,x))) = F(\Phi(g.(t,x))), \qquad (t,x) \in \mathbb{M}^{n+1}$$

where $g.(t,x) \in \mathbb{M}^n$ denotes the action of g on the vector (t,x). Moreover $\alpha_g \circ \alpha_{g'} = \alpha_{gg'}$. (Implictly I'm considering a scalar field).

Accordingly is also natural to assume that there exists a state ω which is invariant under all the $(\alpha_g)_g$ and that the group $(\alpha_g)_g$ has some mild regularity properties.

With some additional mild assumptions one can then derive the existence of a Hilbert space $\mathscr{H}$, a state vector φ, an strongly continuous unitary representation $(U_g)_g$ of the Poincaré group which leaves the state vector φ invariant and a representation $\pi \in \mathrm{Hom}(\mathscr{A}, \mathscr{B}(\mathscr{H}))$ of the observable algebra on $\mathscr{H}$.

Note that $[F(\Phi(t,x)), G(\Phi(t,y))] = 0$ for all F, G and all $x \neq y$. So I can consider all the collection of all $(\Phi(0,x))_{x \in \mathbb{R}^n}$ for all the points at time zero. They form a commutative subalgebra of observables.

Let us now apply the Euclidean strategy and look for a measure which could deliver such data.

We expect the stochastic process X to be, at each point of time $t \in \mathbb{R}$ a random distribution, i.e. that for each $f \in \mathscr{D}(\mathbb{R}^n)$ we can construct a random variable $X_t(f)$ and

$$X_t(\alpha f + \beta g) = \alpha X_t(f) + \beta X_t(g).$$

That is: we have an infinite-dimensional random process, apart from this nothing much changes to the previous finite dimensional considerations above, in particular RP works just fine.

The goal is the to find RP processes in space-time which are invariant wrt. to time and space translations. (What about Poincaré rotations, i.e. rotations and boosts?) One realises that the Poincaré invariance corresponds to the Euclidean invariance of the stochastic process in $\mathbb{R}^{n+1} \approx \mathbb{R}^d$. That is we want the law of X to be invariant under the full Euclidean group acting on $(t, x) \in \mathbb{R}^d$.

► This justifies our definition of EQFT: i.e. probability measure on distributions over $\mathbb{R}^d$ that is RP and Euclidean invariant (+ other technical regularity assumptions).

Let us look therefore at what is available. We can try to take a process X such that the family $(X_t(f))_{t\in\mathbb{R}, f\in\mathscr{D}(\mathbb{R}^n)}$ is jointly Gaussian and we know that to get a RP process we can look at OU processes.

Let us start assuming that we build $X_t(f)$ as a sum of finitely many independent OU (complex) processes (Y_t^i) as follows

$$X_t(f) = \sum_i \hat{f}(k_i) Y_t^i, \tag{1}$$

where $\hat{f}$ is the Fourier transform of f and $k_i \in \mathbb{R}^n$ are parameters. Precisely we take

$$X_t(f) = \sum_i \operatorname{Re}(\hat{f}(k_i)) Y_t^{1,i} + \operatorname{Im}(\hat{f}(k_i)) Y_t^{2,i}.$$

Then we have to arrange the covariances such that

$$\mathbb{E}[X_t(f) X_0(f)] = \sum_{i,j} \operatorname{Re}(\hat{f}(k_i)) \operatorname{Re}(\hat{f}(k_j)) \underbrace{\mathbb{E}[Y_t^{1,i} Y_t^{1,j}]}_{=\delta_{i,j} c_i e^{-\lambda_i |t|}} + \sum_{i,j} \operatorname{Im}(\hat{f}(k_i)) \operatorname{Im}(\hat{f}(k_j)) \underbrace{\mathbb{E}[Y_t^{2,i} Y_t^{2,j}]}_{=\delta_{i,j} c_i e^{-\lambda_i |t|}}$$

$$+ \sum_{i,j} \operatorname{Re}(\hat{f}(k_i)) \operatorname{Im}(\hat{f}(k_j)) \underbrace{\mathbb{E}[Y_t^{1,i} Y_t^{2,j}]}_{=0} + \sum_{i,j} \operatorname{Im}(\hat{f}(k_i)) \operatorname{Re}(\hat{f}(k_j)) \underbrace{\mathbb{E}[Y_t^{2,i} Y_t^{1,j}]}_{=0}$$

Then the covariance reads:

$$\mathbb{E}[X_t(f) X_0(f)] = \sum_i |\hat{f}(k_i)|^2 c_i e^{-\lambda_i |t|}.$$

Exercise. make this precise.

Note that

$$e^{-\lambda_i |t|} = \pi \int_{\mathbb{R}} \frac{e^{i\omega t}}{\omega^2 + \lambda_i^2} \mathrm{d}\omega.$$

(modulo the right coefficient). So

$$\mathbb{E}[X_t(f) X_0(f)] \propto \int_{\mathbb{R}} \sum_i |\hat{f}(k_i)|^2 c_i \frac{e^{i\omega t}}{\omega^2 + \lambda_i^2} \mathrm{d}\omega$$

Now taking more and more points and appropriate c_i we can enforce convergence to a integral over $k \in \mathbb{R}^n$:

$$\mathbb{E}[X_t(f)X_0(f)] = \int_{\mathbb{R}^n}\int_{\mathbb{R}} |\hat{f}(k)|^2 \frac{e^{i\omega t}}{\omega^2 + \lambda(k)^2} \mathrm{d}k\mathrm{d}\omega$$

(we would need now infinitely many degrees of freedom, i.e. OU processes).

This expression implies that the spatial covariance has the form

$$\mathbb{E}[X_t(f(\cdot + x))X_0(f)] = \int_{\mathbb{R}^n}\int_{\mathbb{R}} |\hat{f}(k)|^2 \frac{e^{i(\omega t + k\cdot x)}}{\omega^2 + \lambda(k)^2} \mathrm{d}k\mathrm{d}\omega$$

so it makes sense to take $\lambda(k)^2 = k^2 + m^2$ for a fixed constant $m > 0$. This give rise to an Euclidean invariant covariance. To see this take $f \to \delta$ so $\hat{f}(k) \to 1$ and observe that one obtains

$$\mathbb{E}[X_t(\delta_x)X_0(\delta_0)] = \int_{\mathbb{R}^d} \frac{e^{ik\cdot(t,x)}}{k^2 + m^2} \mathrm{d}k.$$

The associated process is therefore Euclidean invariant in $\mathbb{R}^d$, since the covariance is. This form of the covariance is constrained by RP and Euclidean invariance.

Definition. *This process is called the Gaussian Free field with mass $m > 0$ in (Euclidean) dimension d. This this an example of EQFT.*

For $d = 1$ is just the OU process we have seen.

In particular note that we have problems… :

$$\mathbb{E}[|X_0(\delta_0)|^2] = \int_{\mathbb{R}^d} \frac{1}{k^2 + m^2} \mathrm{d}k = +\infty,$$

if $d \geqslant 2$. Note that $d = 1$ is just QM… (finitely many degrees of freedom). I cannot evaluate X at any fixed space-time point. It is still true that for any $f \in \mathscr{D}(\mathbb{R}^n)$ the process $(X_t(f))_{t\in\mathbb{R}}$ is a nice and continuous Gaussian process.

Exercise. Prove (or motivate) that the stochastic process $(X_t(f))_{t\in\mathbb{R}, f\in\mathscr{D}(\mathbb{R}^n)}$ satisfies the SDE

$$\mathrm{d}X_t(f) = X_t((m^2 - \Delta)^{1/2} f)\mathrm{d}t + \mathrm{d}B_t(f),$$

where $B_t(f)$ is a BM such that $\mathbb{E}[B_t(f)^2] = t\|f\|^2_{L^2(\mathbb{R}^n)}$. This is a very simple example of an SPDE.

Note that our construction guarantees that it is reflection positive in the time direction. By rotation invariant we have also that it is reflection positive along any direction.

Full Euclidean invariance is important because, within the reconstruction of QM guarantees that the quantum theory is Poincaré invariant.

Exercise. Try to work out some details of the reconstruction in this case, it would be nice to arrive at a point where one can check explicitly the Poincaré invariance of the quantum theory.

On Thursday we discuss how to perturb this process with a potential V as we have done in $d = 1$ today, while trying to preserve Euclidean invariance and RP. And then we start to discuss stochastic quantisation.

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