Lecture 5 | 18.2.2021 | 10:00–12:00 via Zoom

Web page: https://www.iam.uni-bonn.de/abteilung-gubinelli/sq-lectures-milan-ws2021
Recorded lectures: https://uni-bonn.sciebo.de/s/6mTx2gYCfCscfFm

Interacting Euclidean Quantum Fields?

Last time we saw the example of the Gaussian free field (GFF) $\varphi\colon \mathbb{R}^d \to \mathbb{R}$, which is the Gaussian EQFT (i.e. RP and Euclidean invariant) with covariance

$$\mathbb{E}[\varphi(x)\varphi(y)] = \int_{\mathbb{R}^d} \frac{e^{ik\cdot(x-y)}}{k^2+m^2} \mathrm{d}k.$$

understood in a distributional sense (more details next week).

What about non-Gaussian examples. We can try the perturbation approach of the last lectures. We know already that if I call μ the law of the GFF then I hope to be able to introduce a new RP measure as follows

$$\nu^L(\mathrm{d}\varphi) = \frac{1}{Z_L} \exp\left(-\int_{-L}^{L} \tilde{V}(\varphi(x_1,\cdot))\mathrm{d}x_1\right)\mu(\mathrm{d}\varphi)$$

where $x_1 \in \mathbb{R}$ is the Euclidean time and $\tilde{V}\colon \mathbb{R}^{\mathbb{R}^{d-1}} \to \mathbb{R}$. But in this way I completely break the Euclidean invariance, so I need a more "symmetric" way to introduce the perturbation, which looks like the above in each coordinate, and therefore in is lead to consider the perturbation

$$\nu^L(\mathrm{d}\varphi) = \frac{1}{Z_L} \exp\left(-\int_{[-L,L]^d} V(\varphi(x))\mathrm{d}x\right)\mu(\mathrm{d}\varphi) \tag{1}$$

where now $V\colon \mathbb{R} \to \mathbb{R}$. This has the form above in every coordinate.

► Of course we broke the translation invariance and also rotation invariance since the integration domain $\Lambda = [-L, L]^d$ has no such symmetries. Is clear that we cannot take $\Lambda = \mathbb{R}^d$ because otherwise the above expression does make sense anymore.

► Moreover we still have the problem that $\varphi(x)$ really does not make sense unless $d = 1$, indeed we will see that φ is only a random distribution i.e. a random element of $\mathscr{S}'(\mathbb{R}^d)$ for $d \geqslant 2$. Actually its regularity degrades with the space dimension d.

► We will also have the problem of understanding what is $V(\varphi(x))$, i.e. a non-linear function of a distribution.

In order to give a meaning to (1) we can regularize the expression $V(\varphi(x))$ by smoothing out φ, i.e. we choose a smooth function $\rho\colon \mathbb{R}^d \to \mathbb{R}$ (e.g. compactly supported) and such that letting $\rho_\varepsilon(x) := \varepsilon^{-d}\rho(\varepsilon^{-1}x)$ we can define

$$\varphi_\varepsilon(x) = (\rho_\varepsilon * \varphi)(x) = \int_{\mathbb{R}^d} \rho_\varepsilon(x-y)\varphi(y)\mathrm{d}y,$$

which is now a well-defined smooth Gaussian field with covariance

$$\mathbb{E}[\varphi_\varepsilon(x)\varphi_\varepsilon(y)] = \int_{\mathbb{R}^d} \frac{|\hat{\rho}(\varepsilon k)|^2 e^{ik\cdot(x-y)}}{k^2+m^2}\mathrm{d}k$$

which is now given by a convergent integral since $\hat{\rho}$ (the Fourier transform or ρ) has rapid decay.

So now we can define $V(\varphi_\varepsilon(x))$ and also a new measure

$$\nu^{\varepsilon,L}(\mathrm{d}\varphi) = \frac{1}{Z_{\varepsilon,L}}\exp\Big(-\int_{[-L,L]^d} V(\varphi_\varepsilon(x))\mathrm{d}x\Big)\mu(\mathrm{d}\varphi). \tag{2}$$

This approach is ok for getting some results but is not the one we will use. One of the problem is that this measure is not anymore RP (exercise: think why).

One can still arrange to smooth only $d-1$ directions and use RP in the remaining direction, this would be fine.

In the rest of these lectures I rather use another approach which preserves both RP and Euclidean invariance (or at least good approximations of them).

Let $\varepsilon = 2^{-N}$ and $M = 2^{N'}$. Let $\Lambda_\varepsilon = (\varepsilon\mathbb{Z})^d \subseteq \mathbb{R}^d$ the square lattice in dimension d of side lenght ε, $\Lambda_{\varepsilon,M} = \Lambda_\varepsilon \cap \mathbb{T}_M^d = (\varepsilon\mathbb{Z})^d \cap [-M/2, M/2)^d$ a finite box of $(M/\varepsilon+1)^d$ points which we think with periodic boundary conditions in every directions.

We are going to discretize our problem on this domain, i.e. replace $[-L,L]^d$ with $\Lambda_{\varepsilon,M}$.

Some useful notation: Fourier tranform on Λ_ε is defined as

$$\mathscr{F}_\varepsilon f(x) = \varepsilon^d \sum_{x\in\Lambda_\varepsilon} f(x)e^{-2\pi ik\cdot x}, \qquad \mathscr{F}_\varepsilon^{-1}g(x) = \int_{\hat{\Lambda}_\varepsilon} g(k)e^{2\pi ik\cdot x}\mathrm{d}k,$$

with $\hat{\Lambda}_\varepsilon = (\varepsilon^{-1}[-1,1))^d$ the dual of Λ_ε. These definitions can be extended to the finite lattice in a natural way, with $\hat{\Lambda}_{\varepsilon,M} = ((\mathbb{Z}/M) \cap [-\varepsilon^{-1}/2, \varepsilon^{-1}/2))^d \approx \Lambda_{M^{-1},\varepsilon^{-1}}$ and

$$\mathscr{F}_{\varepsilon,M} f(x) = \varepsilon^d \sum_{x\in\Lambda_{\varepsilon,M}} f(x)e^{-2\pi ik\cdot x}, \qquad \mathscr{F}_{\varepsilon,M}^{-1}g(x) = \frac{1}{L^d}\sum_{k\in\hat{\Lambda}_{\varepsilon,M}} g(k)e^{2\pi ik\cdot x}.$$

The measure $\mu^{\varepsilon,M}$ is the law of a family of Gaussian r.v. $(\varphi^{\varepsilon,M}(x))_{x\in\Lambda_{\varepsilon,M}}$ with covariance

$$\mathbb{E}_\mu[\varphi^{\varepsilon,M}(x)\varphi^{\varepsilon,M}(y)] = (m^2-\Delta_\varepsilon)^{-1}(x,y), \qquad x,y\in\Lambda_{\varepsilon,M}$$

where $\Delta_{\varepsilon,M}$ is the discrete Laplacian with periodic boundary conditions, i.e.

$$\Delta_\varepsilon f(x) = \varepsilon^{-2}\sum_{i=1,\ldots,d}(f(x+\varepsilon e_i) - 2f(x) + f(x-\varepsilon e_i)), \qquad x\in\Lambda_{\varepsilon,M}$$

where $(e_i)_{i=1,\ldots,d}$ is the canonical basis of $\mathbb{R}^d$. We introduce also discrete derivatives

$$\nabla_\varepsilon^i f(x) = \frac{f(x+\varepsilon e_i) - f(x)}{\varepsilon}, \qquad \nabla_\varepsilon^{-,i} f(x) = \frac{f(x) - f(x-\varepsilon e_i)}{\varepsilon}$$

and note that $(\nabla_\varepsilon^i)^* = -\nabla_\varepsilon^{-,i}$ and $\Delta_\varepsilon = \sum_{i=1}^d \nabla_\varepsilon^{-,i}\nabla_\varepsilon^i$. Moreover

$$(\nabla_\varepsilon^i\mathscr{F}^{-1}g)(x) = \int_{\hat{\Lambda}_\varepsilon} g(k)\frac{e^{2\pi i\varepsilon k_i}-1}{\varepsilon}e^{2\pi ik\cdot x}\mathrm{d}k$$

$$
\begin{aligned}
(\nabla_\varepsilon^{-,i}\nabla_\varepsilon^i \mathscr{F}^{-1} g)(x) &= \int_{\hat{\Lambda}_\varepsilon} g(k) \frac{e^{2\pi i \varepsilon k_i}-1}{\varepsilon}\frac{1-e^{-2\pi i\varepsilon k_i}}{\varepsilon} e^{2\pi i k\cdot x}\mathrm{d}k \\
&= -\int_{\hat{\Lambda}_\varepsilon} g(k)(2\varepsilon^{-1}\sin(\pi\varepsilon k_i))^2 e^{2\pi i k\cdot x}\mathrm{d}k.
\end{aligned}
$$

A Fourier transform formula for the correlation function reads

$$
(m^2-\Delta_\varepsilon)^{-1}(x,y) := \underbrace{\frac{1}{M^d}\sum_{k\in((\mathbb{Z}/M)\cap[-\varepsilon^{-1},\varepsilon^{-1}))^d}}_{\approx\int_{\mathbb{R}^d}} \frac{e^{ik\cdot(x-y)}}{\underbrace{(m^2+\sum_i (2\varepsilon^{-1}\sin(\pi\varepsilon k_i))^2)}_{\approx k^2+m^2}}, \qquad x,y\in\Lambda_{\varepsilon,M}.
$$

So $\varphi^{\varepsilon,M}$ is an approximation of the GFF φ. We denote $\mu^{\varepsilon,M}$ its law, note that it is a law on $\mathbb{R}^{\Lambda_{\varepsilon,M}}$ which is a finite dimensional space. By abuse of notation I will also consider it as a measure on $\mathbb{R}^{\Lambda_\varepsilon}$ by periodic extension.

Both our discrete versions of translation invariance and RP will converge nicely to their continuum counterpart.

► Finally define the measure $\nu^{\varepsilon,M}$ on $\mathbb{R}^{\Lambda_{\varepsilon,M}}$ (or by extension on $\mathbb{R}^{\Lambda_\varepsilon}$)

$$
\nu^{\varepsilon,M}(\varphi) = \frac{1}{Z_{\varepsilon,M}}\exp\left(-\varepsilon^d \sum_{x\in\Lambda_{\varepsilon,M}} V(\varphi(x))\right)\mu^{\varepsilon,M}(\mathrm{d}\varphi) \tag{3}
$$

for some $V:\mathbb{R}\to\mathbb{R}$ bounded below.

Exercise. Prove that if $V(\varphi)=\beta\varphi^2$ and $\beta>-m^2$ then we get another GFF with a different mass.

This approximation now is elementary and it has the advantage that it preserves discrete translation invariance wrt. the lattice Λ_ε and moreover a discrete and periodic version of RP.

Reference for discrete RP: S. Friedli and Y. Velenik, *Statistical Mechanics of Lattice Systems: A Concrete Mathematical Introduction* (Cambridge, United Kingdom; New York, NY: Cambridge University Press, 2017).

► Both our discrete versions of translation invariance and RP will converge nicely to their continuum counterpart as $\varepsilon\to 0$ (to get rid of discreteness) and $M\to\infty$ (to get rid of periodicity).

The rest of the lectures will concern the analysis of these measures in order to prove the existence of the limits above.

What is a proper choice of V? Any V (non-quadratic) is ok, as soon as it works. The problem is that not so many choices are available. In $d=1$ one could take any $V\in C(\mathbb{R},\mathbb{R}_+)$ (or even unbounded with some conditions). In $d=2$ one can take polynomial functions, exponential, trigonometric functions. In $d=3$ we know only how to take V a fourth order polynomial bounded below, in this case we say we are looking at Φ_3^4.

Definition 1. *A Φ_3^4 measure is any non-Gaussian, Euclidean invariant and RP accumulation point of the family $(\nu^{\varepsilon,M})_{\varepsilon,M}$ as $\varepsilon\to 0$ and $M\to\infty$ where one can take as V any 4-th order polinomial, bounded below and with ε,M dependent coefficient.*

One of the big successes of constructive EQFT in '70,'80 is the proof that this limits exists and has many nice properties. It was proven by Glimm, Jaffe, Feldman, Osterwalder, Seneor, …

For ε fixed and $M \to \infty$ this is a problem of statistical mechanics: the infinite volume limit of a system of unbounded spins with nearest neighbor interaction.

(this ends the first part of the course, i.e. the passage QM to EQFT and the formulation of our main problem)

Second part of the lectures: Stochastic quantisation

As I said at the beginning a stochastic quantisation (in these lectures) of a given measure ρ is a map F_ρ which sends a Gaussian r.v. to a r.v. with law ρ.

Even in the case $\nu^{\varepsilon,M}$ there are many interesting ways to do this.

1. *Langevin dynamics* / parabolic SQ: the Gaussian process W is a family of Brownian motions and the map $\nu \sim F_\nu(W) = \phi(0)$ is given by the stationary solution

   $$\phi: \mathbb{R} \times \Lambda_{\varepsilon,M} \to \mathbb{R},$$

   of the SDE

   $$d\phi(t,x) = \{[(-m^2 + \Delta_\varepsilon)\phi(t)](x) - V'(\phi(t,x))\}\mathrm{d}t + \mathrm{d}W(t,x), \qquad x \in \Lambda_{\varepsilon,M}, t \in \mathbb{R}$$

   with V' the derivative of V. Here $t \in \mathbb{R}$ is a fictious time (it is not the Euclidean time!!!)

2. *Elliptic SQ*: $\nu \sim F_\nu(\xi) = (\phi(0,x))_{x \in \Lambda_{\varepsilon,M}}$ but now $\phi: \mathbb{R}^2 \times \Lambda_{\varepsilon,M} \to \mathbb{R}$ is the solution to the elliptic PDE

   $$(m^2 - \Delta_{\mathbb{R}^2} - \Delta_{\Lambda_{\varepsilon,M}})\phi(z,x) + V'(\phi(z,x)) = \xi(z,x), \qquad x \in \Lambda_{\varepsilon,M}, z \in \mathbb{R}^2$$

   where ξ is a space-time white noise.

3. *Canonical SQ*: the Gaussian process W is a family of Brownian motions and the map $\nu \sim F_\nu(W) = \phi(0)$ is given by the stationary solution

   $$\phi: \mathbb{R} \times \Lambda_{\varepsilon,M} \to \mathbb{R}$$

   of the SDE (discrete wave equation)

   $$\partial_t^2\phi(t,x) = -\gamma\,\partial_t\phi(t,x) + [(-m^2 + \Delta_\varepsilon)\phi(t)](x) - V'(\phi(t,x)) + \partial_t W(t,x)$$

   (approximatively). Without noise this is an Hamiltonian equation.

4. *Variational representation* (see Barashkov/G.)

5. There is even another possible approach which require to consider a stochastic evolution in the Euclidean time and looks like

   $$\partial_{x_0}\phi = \{-(m^2 - \Delta_\varepsilon)^{1/2}\phi - V'(\phi)\}\mathrm{d}t + \partial_{x_0}W, \qquad x \in \mathbb{R} \times \Lambda_{\varepsilon,M}^{d-1}$$

in this case we cannot discretize the Euclidan time and also the measure $\nu^{\varepsilon,M}$ has to be taken slightly differently. This is essentially the Markovian point of view wrt. the EQFT where we perturb the OU process ϕ with a drift $-V'(\phi)$.

Remark. While the measure $\nu^{\varepsilon,M}$ is defined via a density wrt. to a Gaussian the goal of SQ is to define it as the pushforward of a Gaussian measure. In infinite dimensions it seems that pushforward are more robust.

Example. Let $(B_t)_{t\geqslant 0}$ a one dim BM and let $X_t = B_t + t$. Then while there is no problem to see the law of X as pushforward of that of B, they are not absolutely continuous.

Exercise. Prove it.

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