Lecture 6 | 18.2.2021 | 14:00–16:00 via Zoom

Web page: https://www.iam.uni-bonn.de/abteilung-gubinelli/sq-lectures-milan-ws2021
Recorded lectures: https://uni-bonn.sciebo.de/s/6mTx2gYCfCscfFm

Some notations from this morning.

The measure $\mu^{\varepsilon,M}$ is the law of a family of Gaussian r.v. $(\varphi^{\varepsilon,M}(x))_{x\in\Lambda_{\varepsilon,M}}$ with covariance

$$\mathbb{E}_\mu[\varphi^{\varepsilon,M}(x)\varphi^{\varepsilon,M}(y)] = (m^2-\Delta_\varepsilon)^{-1}(x,y), \qquad x,y\in\Lambda_{\varepsilon,M}$$

The measure $\nu^{\varepsilon,M}$ on $\mathbb{R}^{\Lambda_{\varepsilon,M}}$ (or by extension on $\mathbb{R}^{\Lambda_\varepsilon}$) is defined as

$$\nu^{\varepsilon,M}(\varphi) = \frac{1}{Z_{\varepsilon,M}}\exp\left(-\varepsilon^d\sum_{x\in\Lambda_{\varepsilon,M}} V(\varphi(x))\right)\mu^{\varepsilon,M}(\mathrm{d}\varphi) \tag{1}$$

for some $V:\mathbb{R}\to\mathbb{R}$ bounded below. We will soon take $V(\varphi) = \lambda\varphi^4+\beta\varphi^2$ with $\lambda>0$ and $\beta\in\mathbb{R}$ (or just $\beta=0$).

References: stochastic quantisation was introduced Nelson, Parisi & Wu. Rigorous construction of EQFT with stochastic quantisation was done in $d=1$ by Jona–Lasinio and Faris ('80), Jona–Lasinio and Mitter (~'84) in $d=2$ bounded volume and then Mitter et al. in infinite volume, this was done using probabilistic tools (martingale problems and Girsanov's formula). For $P(\Phi)_2$ (polynomial interaction in $d=2$) another approach was introduced by Da Prato and Debussche. Only in 2013 Hairer managed to prove a local existence and uniqueness result for the parabolic SQ of Φ^4_3 using regularity structures. And the we had many more results… still many problems remain open.

For more details on the history of EQFT and SQ look at the introductions of these three papers:

- M. Gubinelli and M. Hofmanova, `A PDE Construction of the Euclidean Φ^4_3 Quantum Field Theory', *ArXiv:1810.01700 [Math-Ph]*, 3 October 2018, http://arxiv.org/abs/1810.01700.
- S. Albeverio, F. C. De Vecchi, and Massimiliano Gubinelli, `Elliptic Stochastic Quantization', *Annals of Probability* 48, no. 4 (July 2020): 1693–1741, https://doi.org/10.1214/19-AOP1404.
- S. Albeverio et al., `Grassmannian Stochastic Analysis and the Stochastic Quantization of Euclidean Fermions', *ArXiv:2004.09637 [Math-Ph]*, 25 May 2020, http://arxiv.org/abs/2004.09637.

For hyperbolic SQ and the variational method one could refer to

- M. Gubinelli, H. Koch, and T. Oh, `Renormalization of the Two-Dimensional Stochastic Nonlinear Wave Equations', *Transactions of the American Mathematical Society*, 2018, 1, https://doi.org/10.1090/tran/7452.
- N. Barashkov and M. Gubinelli, `A Variational Method for Φ^4_3 ', *Duke Mathematical Journal* 169, no. 17 (November 2020): 3339–3415, https://doi.org/10.1215/00127094-2020-0029.

Langevin dynamics

We start by constructing the parabolic stochastic quantisation of the measure $\nu^{\varepsilon,M}$ for fixed ε, M. Since in this section these parameters do not play any role we will avoid to write the whenever it does not lead to ambiguities. In particular here Λ will denote the finite set $\Lambda_{\varepsilon,M}$ and Δ the discrete Laplacian.

The law $\mu^{\varepsilon,M}$ is Gaussian, we can therefore introduce a fictious time $t \in \mathbb{R}$ (this is !not! the physical time) and a stationary OU process $(X_t^{\varepsilon,M})_{t \geqslant 0}$ such that $X_t^{\varepsilon,M} \sim \mu^{\varepsilon,M}$. There is not a unique choice, however it is not difficult to guess that a suitable dynamics is given by

$$\mathrm{d}X_t^{\varepsilon,M} = (\Delta_\varepsilon - m^2) X_t^{\varepsilon,M} \mathrm{d}t + 2^{1/2} \mathrm{d}B_t^{\varepsilon,M}, \tag{2}$$

where $(B_t^{\varepsilon,M}(x))_{x \in \Lambda_{\varepsilon,M}}$ is a family of independent standard Brownian motions.

Exercise. Check the invariance of $\mu^{\varepsilon,M}$ under this dynamics, in particular pay attention to the normalization.

I want to construct now a dynamics which leave invariant the measure $\nu^{\varepsilon,M}$ instead.

Let us guess what this dynamics shoud be: we write something similar as what we had before but with an unknown vectorfield $F(t)$

$$\mathrm{d}X_t = A X_t \mathrm{d}t + F(t) \mathrm{d}t + 2^{1/2} \mathrm{d}B_t.$$

with $A = (\Delta - m^2)$. Then if we denote $\mathbb{P}$ the law of the solution X of this equation with $X_0 \sim \mu^{\varepsilon,M}$ and independent B, we want to have

$$\int_{\mathbb{R}^\Lambda} f(\varphi) \nu(\mathrm{d}\varphi) = \int_{\mathbb{R}^\Lambda} f(\varphi) \frac{e^{-U(\varphi)}}{Z} \mu(\mathrm{d}\varphi) = \frac{1}{Z} \mathbb{E}[f(X_0) e^{-U(X_0)}] = \frac{1}{Z} \mathbb{E}[f(X_t) e^{-U(X_0)}],$$

for all test functions f and all $t \geqslant 0$ with

$$U(\varphi) = \varepsilon^d \sum_{x \in \Lambda_{\varepsilon,M}} V(\varphi(x)), \qquad U \colon \mathbb{R}^{\Lambda_{\varepsilon,M}} \to \mathbb{R}.$$

Note that under the measure $\mathbb{P}^U$ defined as

$$\mathbb{P}^U(\mathrm{d}X) = \frac{e^{-U(X_0)}}{Z} \mathbb{P}(\mathrm{d}X)$$

the process X is still solution to the equation and $X_0 \sim \nu^{\varepsilon,M}$.

► By Girsanov's formula, we have, for two test functions f, g

$$\mathbb{E}[f(X_t) e^{-U(X_0)} g(X_0)] = \mathbb{E}_{\mathbb{Q}}\Big[f(X_t) e^{\int_0^t F(s) 2^{1/2} \mathrm{d}W_s - \int_0^t |F(s)|^2 \mathrm{d}s - U(X_0)} g(X_0)\Big] \tag{3}$$

where under $\mathbb{Q}$ the process X satisfy the linear SDE

$$\mathrm{d}X_t = A X_t \mathrm{d}t + 2^{1/2} \mathrm{d}W_t$$

where W is a BM under $\mathbb{Q}$. Note that $X_0 \sim \mu$. So under $\mathbb{Q}$ X is an stationary OU process.

Note that Ito formula gives

$$
\begin{aligned}
U(X_t) &= U(X_0) + \int_0^t \mathrm{D}U(X_s)\mathrm{d}X_s + \int_0^t D^2U(X_s)\mathrm{d}s \\
&= U(X_0) + \int_0^t \mathrm{D}U(X_s)2^{1/2}\mathrm{d}W_s + \int_0^t \underbrace{(\mathrm{D}^2U(X_s) + \mathrm{DU}(X_s)AX_s)}_{=:Q(X_s)}\mathrm{d}s
\end{aligned}
$$

so we can rewrite (3) as

$$
= \mathbb{E}_{\mathbb{Q}}\left[f(X_t)g(X_0)e^{-\frac{1}{2}U(X_t)-\frac{1}{2}U(X_0)+\frac{1}{2}\int_0^t(Q(X_s)\mathrm{d}s-|F(s)|^2)\mathrm{d}s}e^{\int_0^t F(s)2^{1/2}\mathrm{d}W_s+\frac{1}{2}\int_0^t \mathrm{D}U(X_s)2^{1/2}\mathrm{d}W_s}\right]
$$

and then take $F(s) = -\frac{1}{2}\mathrm{D}U(X_s)$ to cancel the stochastic integral in the exponent to get

$$
\mathbb{E}[f(X_t)e^{-U(X_0)}g(X_0)] = \mathbb{E}_{\mathbb{Q}}\left[f(X_t)g(X_0)e^{-\frac{1}{2}U(X_t)-\frac{1}{2}U(X_0)+\frac{1}{2}\int_0^t\left(Q(X_s)\mathrm{d}s-\frac{1}{4}|\mathrm{D}U(X_s)|^2\right)\mathrm{d}s}\right]
$$

and since $\mathbb{Q}$ is time reflection invariant (because under $\mathbb{Q}$ the process X is just a stationary OU process) we can rewrite this as

$$
= \mathbb{E}_{\mathbb{Q}}\left[f(X_0)g(X_t)e^{-\frac{1}{2}U(X_t)-\frac{1}{2}U(X_0)+\frac{1}{2}\int_0^t\left(Q(X_s)\mathrm{d}s-\frac{1}{4}|\mathrm{D}U(X_s)|^2\right)\mathrm{d}s}\right]
$$

where we exchanged the two functions. Taking $f = 1$ we have

$$
\mathbb{E}_{\mathbb{Q}}\left[g(X_t)e^{-\frac{1}{2}U(X_t)-\frac{1}{2}U(X_0)+\frac{1}{2}\int_0^t\left(Q(X_s)\mathrm{d}s-\frac{1}{4}|\mathrm{D}U(X_s)|^2\right)\mathrm{d}s}\right] = \mathbb{E}[e^{-U(X_0)}g(X_0)]
$$

and on the other hand, taking $g = 1$ we have (taking $g = f$ in the previous formula)

$$
\mathbb{E}[f(X_t)e^{-U(X_0)}] = \mathbb{E}_{\mathbb{Q}}\left[f(X_t)e^{-\frac{1}{2}U(X_t)-\frac{1}{2}U(X_0)+\frac{1}{2}\int_0^t\left(Q(X_s)\mathrm{d}s-\frac{1}{4}|\mathrm{D}U(X_s)|^2\right)\mathrm{d}s}\right] = \mathbb{E}[e^{-U(X_0)}f(X_0)]
$$

that is what we were looking for.

Remark. All this is ok provided we can perform all these computations. The only problems are related to the integrability of the exponential function involving the time integral. For example if we require that U is bounded below and moreover that

$$
H(\varphi) = \frac{1}{2}Q(\varphi) - \frac{1}{8}|\nabla U(\varphi)|^2 = \mathrm{D}^2U(\varphi) + \mathrm{DU}(\varphi)\cdot A\varphi - |\mathrm{D}U(\varphi)|^2
$$

satisfies

$$
\mathbb{E}_{\mathbb{Q}}\left[e^{\int_0^t H(X_s)\mathrm{d}s}\right] < \infty,
$$

for some $t > 0$. Indeed it is enough to establish invariance for small time and the for all times.

► We learned that the solution to

$$
\mathrm{d}X_t = AX_t\mathrm{d}t - \frac{1}{2}\mathrm{D}U(X_t)\mathrm{d}t + 2^{1/2}\mathrm{d}B_t \tag{4}
$$

leaves the measure $\nu(\mathrm{d}\varphi) = Z^{-1}e^{-U(\varphi)}\mu(\mathrm{d}\varphi)$ invariant provided U is nice enough.

We will take this equation as stochastic quantisation.

Exercise. Note that the process X is time-reversal invariant (we essentially gave a proof of this above, you can fill in the details).

We would actually like to have U which are unbounded, but bounded below, the relevant example in these lectures being

$$U(\varphi) = \varepsilon^d \sum_x \left(\frac{\lambda}{4}\varphi(x)^4 + \frac{\beta}{2}\varphi(x)^2 \right)$$

for some $\lambda > 0$ and $\beta \in \mathbb{R}$.

We have two order of problems with such potentials. First

$$\mathrm{D}_x U(\varphi) = \frac{\partial}{\partial \varphi(x)} U(\varphi) = \lambda \varphi(x)^3 + \beta \varphi(x)$$

is not globally Lipschitz and the solutions to the SDE (4) could explode in finite time.

Then we still have to worry about invariance (i.e. fixing the details of the argument above) of the measure ν on this dynamics.

The second problem is merely technical and could be handled via a careful control of approximations with nice U and the above invariance argument. The first problem seems more worrisome but the key is to exploit the coercivity of the dynamics.

First method: one could use the invariance of the measure ν to conclude that solutions of the SDE do not explode, we will not do it here.

Second method: A direct approach is to test the equation with X_t, i.e. write by Ito formula

$$\frac{1}{2}\mathrm{d}_t \sum_x |X_t(x)|^2 = \sum_x X_t(x)\mathrm{d}X_t(x) + \sum_x \mathrm{d}t$$

$$= \sum_x \left[X_t(x)(AX_t)(x) - \frac{1}{2}X_t(x)\mathrm{D}_x U(X_t) \right]\mathrm{d}t + 2^{1/2}\sum_x X_t(x)\mathrm{d}B_t(x) + \sum_x \mathrm{d}t$$

$$= -G(X_t)\mathrm{d}t + \beta \sum_x X_t(x)^2 \mathrm{d}t + 2^{1/2}\sum_x X_t(x)\mathrm{d}B_t(x) + \sum_x \mathrm{d}t$$

with in the polynomial case (summing by parts the Laplacian)

$$G(\varphi) = \sum_x (|\nabla_\varepsilon \varphi(x)|^2 + m^2 \varphi(x)^2 + \lambda \varphi(x)^4) \geqslant 0.$$

By taking averages we could get some interesting estimates, for example

$$\mathbb{E}\sum_x |X_t(x)|^2 + \int_0^t G(X_s)\mathrm{d}s = \beta \int_0^t \mathbb{E}\sum_x X_s(x)^2 \mathrm{d}s + \sum_x \mathrm{d}t,$$

where now the r.h.s. can be controlled via the l.h.s. or via Gronwall lemma.

But this is not robust enough for what is going on next week.

Third and last method: a more elementary and useful in the following approach which do not rely on Ito's formula goes as follows (this essentially what is called the Da Prato–Debussche trick).

First one write $X = Y + Z$ where Y is the solution to the linear equation

$$\mathrm{d}Y_t = AY_t\mathrm{d}t + 2^{1/2}\mathrm{d}B_t$$

that is an OU process, and Z is what remains.

Then Z must solve

$$\frac{\mathrm{d}Z_t}{\mathrm{d}t} = (AZ_t - \nabla U(Y_t + Z_t))$$

which is an ODE with random coefficients, not a stochastic differential equation anymore since the effect of the Brownian perturbation is completely taken into account by Y.

We can now test this equation with Z (without the need of Ito's formula) and obtain

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\sum_x |Z_t(x)|^2 + G(Z_t) = -\lambda \sum_x (Y_t(x)^3 Z_t(x) + 3Y_t(x)^2 Z_t(x)^2 + 3Y_t(x) Z_t(x)^3)$$

$$+\beta \sum_x (Z_t(x) Y_t(x) + Z_t(x)^2)$$

where

$$G(\varphi) = \|\nabla \varphi\|_{L^2}^2 + m^2 \|\varphi\|_{L^2}^2 + \lambda \|\varphi\|_{L^4}^4$$

with the natural Lebesgue spaces on $\Lambda = \Lambda_{\varepsilon,M}$ (with counting measure).

The key property being that in the r.h.s. we have all terms which we can bound via Hölder inequality as

$$\frac{\mathrm{d}}{\mathrm{d}t}\sum_x |Z_t(x)|^2 + G(Z_t) \leqslant C_\delta \|Y_t\|_{L^4}^4 + \delta G(Z_t)$$

for $\delta > 0$ small as we wish, e.g. $\delta = 1/2$. We conclude that

$$\|Z_t\|_{L^2}^2 + \frac{1}{2}\int_0^t G(Z_s)\mathrm{d}s \leqslant \|Z_0\|_{L^2}^2 + C\int_0^t \|Y_s\|_{L^4}^4 \mathrm{d}s. \quad (5)$$

This bound implies that solutions cannot explode and we have an explicit bound on its growth in term of Y and Z_0. Of the two we know very well Y (it is the OU process, it is Gaussian, I know everything I want on it). On the other hand I do not know so well

$$Z_0 = X_0 - Y_0 \sim \text{“}\nu - \mu\text{”}$$

because we do not really know very well ν (which is actually the object we want to study). For example we do not know estimates uniform in ε, M.

Note that even if X is stationary and Y is stationary (because we take $X_0 \sim \nu$ and $Y_0 \sim \mu$ and independent). But they are not indenepend and more importantly Z is not stationary.

One would like to prove that there exists a coupling of X_0 and Y_0 (i.e. find a joint law with marginals ν and μ respectively) so that the process (X, Y) is stationary (as a pair) from which would follow that Z is stationary.

In any case what we have so far is that for any f and any t we have

$$\int f(\varphi)\nu(\mathrm{d}\phi) = \mathbb{E}[f(X_t)] = \frac{1}{t}\int_0^t \mathbb{E}[f(X_s)]\mathrm{d}s = = \frac{1}{t}\int_0^t \mathbb{E}[f(Y_s + Z_s)]\mathrm{d}s$$

using stationarity. This is the stochastic quantization equation. Estimates on X are given via Y and Z.

Let's construct a stationary coupling of Y and Z. One uses the Krylov-Bogoliubov argument. We can construct a measure γ_T on a pair of fields $(\varphi, \psi) \in \mathbb{R}^\Lambda \times \mathbb{R}^\Lambda$ by the formula

$$\int f(\varphi, \psi) \mathrm{d}\gamma_T(\varphi, \psi) := \frac{1}{T}\int_0^T \mathbb{E}[f(Y_s, Z_s)]\mathrm{d}s,$$

for any bounded function f of the pair $(\varphi, \psi) \in \mathbb{R}^\Lambda \times \mathbb{R}^\Lambda$ where Y, Z are started as above.

We have that

$$\int [G(\psi) + \|\varphi\|_{L^4}^4]\mathrm{d}\gamma_T(\varphi, \psi) = \frac{1}{T}\int_0^T \mathbb{E}[G(Z_s) + \|Y_s\|_{L^4}^4]\mathrm{d}s \leqslant \frac{2}{T}\Big(\mathbb{E}\|Z_0\|_{L^2}^2 + C'\int_0^T \mathbb{E}\|Y_s\|_{L^4}^4 \mathrm{d}s\Big),$$

$$\leqslant \Big(\frac{2}{T}\mathbb{E}\|Z_0\|_{L^2}^2\Big) + 2C'\mathbb{E}\|Y_0\|_{L^4}^4,$$

which is uniformly bounded in T. This implies that the family $(\gamma_T)_T$ is tight on $\mathbb{R}^\Lambda \times \mathbb{R}^\Lambda$ and one can extract a weakly convergent subsequence to a limit γ.

Note also that

$$\int f(\varphi + \psi)\mathrm{d}\gamma_T(\varphi, \psi) = \frac{1}{T}\int_0^T \mathbb{E}[f(Y_s + Z_s)]\mathrm{d}s = \frac{1}{T}\int_0^T \mathbb{E}[f(X_s)]\mathrm{d}s = \mathbb{E}[f(X_0)] = \int f(\varphi)\nu(\mathrm{d}\phi)$$

Therefore the law of $\varphi + \psi$ under γ_T is always given by ν for any T. As a consequence the law of $\varphi + \psi$ under γ is ν.

The measure γ is stationary under the joint dynamics of (Z, Y), i.e. if $(Z_0, Y_0) \sim \gamma$ then $(Z_t, Y_t) \sim \gamma$.

Exercise. Prove it. Also try to understand if the dynamics of the pair is time-symmetric.

In this way one can construct a stationary coupling of (Z, Y) which gives a useful representation of the stationary process X.

Infinite volume limit

What happens when we want to take the limit $M \to \infty$? The estimate (5) is not good enough because both $\|Z_0\|_{L^2(\Lambda_{\varepsilon,M})}$ and $\|Y_s\|_{L^4(\Lambda_{\varepsilon,M})}$ cannot remain finite since both random field are stationary and one expects that

$$\|Z_0\|_{L^2(\Lambda_{\varepsilon,M})} \sim M^d, \qquad \|Y_s\|_{L^4(\Lambda_{\varepsilon,M})} \sim M^d.$$

In this section we explicit the dependence on M and use Λ_ε for the full lattice. Moreover we extend any periodic field to the full lattice periodically. (we fix an origin)

However we can modify our apriori estimate introducing a polynomial weight $\rho: \Lambda = (\varepsilon\mathbb{Z})^d \to \mathbb{R}$

$$\rho(x) = (1 + \ell|x|)^{-\sigma}$$

with $\ell, \sigma > 0$ large enough, where $|x|$ is the distance from the origin of $\Lambda = \Lambda_\varepsilon$.

Now we test the equation for Z with $\rho^2 Z$ summing over the full lattice Λ and we get

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\sum_{x \in \Lambda_\varepsilon} |\rho(x)Z_t(x)|^2 + G(Z_t) \leqslant -\lambda \sum_{x \in \Lambda_\varepsilon} \rho(x)(Y_t(x)^3 Z_t(x) + 3Y_t(x)^2 Z_t(x)^2 + 3Y_t(x)Z_t(x)^3)$$

$$+\beta \sum_{x \in \Lambda_\varepsilon} \rho(x)(Z_t(x)Y_t(x) + Z_t(x)^2) + C_\rho \sum_{x \in \Lambda_\varepsilon} \rho(x)Z_t(x)^2$$

where C_ρ (and the inequality) is term coming from the integration by parts which can be made small by choosing ℓ small and where

$$G(\varphi) = \|\rho\nabla\varphi\|_{L^2(\Lambda_\varepsilon)}^2 + m^2\|\rho\varphi\|_{L^2(\Lambda_\varepsilon)}^2 + \lambda\|\rho^{1/2}\varphi\|_{L^4(\Lambda_\varepsilon)}^4.$$

And using similar estimates as above we obtain the apriori weighted estimates:

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\rho Z_t\|_{L^2(\Lambda_\varepsilon)}^2 + G(Z_t) \leqslant C_\delta\|\rho^{1/2}Y_t\|_{L^4(\Lambda_\varepsilon)}^4 + \delta G(Z_t)$$

indeed

$$\lambda\left|\sum_{x\in\Lambda_\varepsilon}\rho(x)Y_t(x)^3Z_t(x)\right| \leqslant \lambda\left|\sum_{x\in\Lambda_\varepsilon}(\rho(x)^{3/2}Y_t(x)^3)(\rho(x)^{1/2}Z_t(x))\right|$$

$$\leqslant \lambda\frac{C}{\delta}\|\rho^{1/2}Y_t\|_{L^4}^4 + \delta\lambda\|\rho^{1/2}Z_t\|_{L^4}^4 \leqslant \lambda\frac{C}{\delta}\|\rho^{1/2}Y_t\|_{L^4}^4 + \delta G(Z_t)$$

for any small $\delta > 0$.

As a consequence one get the estimate

$$\|\rho Z_t\|_{L^2(\Lambda)}^2 + \frac{1}{2}\int_0^t G(Z_s)\mathrm{d}s \leqslant \|\rho Z_0\|_{L^2(\Lambda)}^2 + C\int_0^t \|\rho^{1/2}Y_s\|_{L^4(\Lambda)}^4\mathrm{d}s. \tag{6}$$

We have seen that we can construct a statiorary coupling of (Y, Z), so we can use there this stationary coupling and take the average of this inequality to get

$$\mathbb{E}\|\rho Z_t\|_{L^2(\Lambda)}^2 + \frac{1}{2}\int_0^t \mathbb{E}G(Z_s)\mathrm{d}s \leqslant \mathbb{E}\|\rho Z_0\|_{L^2(\Lambda)}^2 + C\int_0^t \mathbb{E}\|\rho^{1/2}Y_s\|_{L^4(\Lambda)}^4\mathrm{d}s$$

but by stationarity we also have $\mathbb{E}\|\rho Z_t\|_{L^2(\Lambda)}^2 = \mathbb{E}\|\rho Z_0\|_{L^2(\Lambda)}^2$ so the initial condition disappear!!! So

$$\frac{1}{2}\int_0^t \mathbb{E}G(Z_s)\mathrm{d}s \leqslant C\int_0^t \mathbb{E}\|\rho^{1/2}Y_s\|_{L^4(\Lambda)}^4\mathrm{d}s$$

and again by stationarity one get

$$\mathbb{E}G(Z_0) \leqslant 2C\mathbb{E}\|\rho^{1/2}Y_0\|_{L^4(\Lambda)}^4.$$

Which give us very good apriori estimates on the law of Z_0 which are independent of M, indeed

$$\mathbb{E}\|\rho^{1/2}Y_0\|_{L^4(\Lambda)}^4 = \mathbb{E}\sum_{x\in\Lambda_\varepsilon}\rho(x)^2|Y_0(x)|^4 = \sum_{x\in\Lambda_\varepsilon}\rho(x)^2\mathbb{E}|Y_0(x)|^4 = C\sum_{x\in\Lambda_\varepsilon}\rho(x)^2 < \infty$$

uniformly in M provided $\sigma > d$ and the law of $Y_0(x)$ is translation invariant so does not depend on x and actually one can easily show that

$$(\mathbb{E}|Y_0(x)|^4)^{1/2} \leqslant C\mathbb{E}|Y_0(x)|^2$$

$$\lesssim (m^2 - \Delta)^{-1}(x, x) \lesssim \frac{1}{M^d}\sum_{k\in((\mathbb{Z}/M)\cap[-\varepsilon^{-1},\varepsilon^{-1}))^d}\frac{1}{(m^2 + \sum_i (2\varepsilon^{-1}\sin(\pi\varepsilon k_i))^2)}$$

$$\to \int_{[-\varepsilon^{-1},\varepsilon^{-1})^d}\frac{1}{(m^2 + \sum_i (2\varepsilon^{-1}\sin(\pi\varepsilon k_i))^2)} < +\infty$$

uniformly in M.

Lemma 1. *For any $M>0$ we have that $\nu^M \sim X_0^M \sim Y_0^M + Z_0^M$ where $Y_0^M \sim \mu^M$ and Z_0^M is a r.v. such that*

$$\sup_M \mathbb{E} G(Z_0^M) < \infty.$$

This is a key estimate to take the infinite volume limit since it allows to use tightness on the family $(\nu^M)_M$ in the topology of local convergence.

These lecture notes are produced using the computer program $\text{T}_\text{E}\text{X}_{\text{MACS}}$. If you want to know more go here `www.texmacs.org`.