Lecture 7 | 22.2.2021 | 10:00–12:00 via Zoom

Web page: https://www.iam.uni-bonn.de/abteilung-gubinelli/sq-lectures-milan-ws2021
Recorded lectures: https://uni-bonn.sciebo.de/s/6mTx2gYCfCscfFm

Some notions from last week.

We have a measure $\mu^{\varepsilon,M}$ given by the law of a family of Gaussian r.v. $(\varphi^{\varepsilon,M}(x))_{x\in\Lambda_{\varepsilon,M}}$ with covariance

$$\mathbb{E}_\mu[\varphi^{\varepsilon,M}(x)\varphi^{\varepsilon,M}(y)] = (m^2-\Delta_\varepsilon)^{-1}(x,y), \qquad x,y\in\Lambda_{\varepsilon,M}$$

We introduced a "perturbed" measure $\nu^{\varepsilon,M}$ on $\mathbb{R}^{\Lambda_{\varepsilon,M}}$ (or by extension on $\mathbb{R}^{\Lambda_\varepsilon}$) is defined as

$$\nu^{\varepsilon,M}(\varphi) = \frac{1}{Z_{\varepsilon,M}}\exp\left(-\varepsilon^d\sum_{x\in\Lambda_{\varepsilon,M}} V(\varphi(x))\right)\mu^{\varepsilon,M}(\mathrm{d}\varphi) \tag{1}$$

for some $V:\mathbb{R}\to\mathbb{R}$ bounded below. We will take often

$$V(\varphi) = \lambda\varphi^4 + \beta\varphi^2$$

with $\lambda>0$ and $\beta\in\mathbb{R}$ (or just $\beta=0$).

Remember that as long as $\varepsilon>0$ the quantity $(m^2-\Delta_\varepsilon)^{-1}(x,y)$ is bounded (it is a finite sum).

Last week we was how to take the infinite volume limit: better, how to obtain the suitable weighted estimates which are uniform in M and allow to prove tightness of the family $(\nu^{\varepsilon,M})_M$ for fixed $\varepsilon>0$ and $M\to\infty$, in the topology of local convergence (i.e. convergence by testing with continous functions on $\mathbb{R}^{\Lambda_\varepsilon}$ which depends only of finitely many points of Λ_ε). In particular we understood that the local (or weighted) $L^p(\Lambda_\varepsilon)$ norms of $\varphi:\mathbb{R}^{\Lambda_\varepsilon}\to\mathbb{R}$ under the measure $\nu^{\varepsilon,M}$ have finite moments:

$$\sup_M\int\|\rho\varphi\|_{L^p}^p\nu^{\varepsilon,M}(\mathrm{d}\varphi)<\infty$$

for any $p>1$. Actually by working a bit harder one can prove uniform integrability of functions like $\exp(\|\rho\varphi\|_{L^2})$.

What about uniqueness of the accumulation points? In the extended notes I will also discuss a bit this issue and using essentially a similar approach one can prove that provided

$$V''(\varphi)\geqslant-\chi,$$

for some $\chi>0$ then for m large enough (depending on χ) we have also uniqueness of the limit measure ν^ε. This is natural because we do not expect in general that the limit is unique (there could be phase transitions in the model, in $d\geqslant2$ since it is a model of ferromagnetic unbounded spin).

The idea to prove uniquess is to compare two solutions Z^1,Z^2 driven by two Gaussian processes Y^1,Y^2 and use a coupling approach.

Today the plan is to address the other problem: that is we keep M fixed (let's say $M=1$) and send $\varepsilon \to 0$. This is the ultraviolet limit (UV limit). Obtain uniform estimates in this limit is more difficult are requires new ideas. There are various possible approaches: regularity structures (Hairer), renormalization group ideas (Kupiainen), or paracontrolled distributions (GIP, Catellier & Chouk). I will follow this last strategy. The main reference for us here is the papere I mentioned by Hofmanova & G. [Massimiliano Gubinelli and Martina Hofmanova, `A PDE Construction of the Euclidean Φ_3^4 Quantum Field Theory', *ArXiv:1810.01700 [Math-Ph]*, 3 October 2018, http://arxiv.org/abs/1810.01700.]

The main problem is that as $\varepsilon \to 0$ the process Y becomes a distribution. Recall our context. We had a dynamics on X which can be decomposed on a linear part

$$\mathrm{d}Y_t = (\Delta_\varepsilon - m^2)\, Y_t \mathrm{d}t + 2^{1/2}\mathrm{d}B_t$$

and the non-linear part Z:

$$\frac{\partial}{\partial t} Z_t = (\Delta_\varepsilon - m^2)\, Z_t - \frac{1}{2} V'(Y_t + Z_t) \qquad (2)$$

with $V'(\varphi) = \lambda \varphi^3 + \beta \varphi$. The computation of $V'(Y_t + Z_t)$ is pointwise in space:

$$V'(Y_t + Z_t)(x) = V'(Y_t(x) + Z_t(x)) = \lambda (Y_t(x) + Z_t(x))^3 + \beta (Y_t(x) + Z_t(x))$$

$$= \lambda Y_t(x)^3 + 3\lambda Y_t(x)^2 Z_t(x) + 3\lambda Y_t(x) Z_t(x)^2 + \lambda Z_t(x)^3 + \beta Y_t(x) + \beta Z_t(x).$$

The main problem is the following: ($M=1$) as $\varepsilon \to 0$

$$\mathbb{E}[Y_t(x)^2] = (m^2 - \Delta_\varepsilon)(x,x) = \sum_{k \in (\mathbb{Z} \cap [-\varepsilon^{-1}, \varepsilon^{-1}))^d} \frac{1}{(m^2 + \sum_i (2\varepsilon^{-1} \sin(\pi \varepsilon k_i))^2)}$$

$$\approx \sum_{k \in \mathbb{Z} \cap [-\varepsilon^{-1}, \varepsilon^{-1})^d} \frac{1}{(m^2 + 2\pi |k|^2)} \propto \begin{cases} \varepsilon^{2-d} & d > 2 \\ \log(\varepsilon^{-1}) & d = 2 \end{cases}$$

Which tells us that the typical size of $Y_t(x)$ is $\varepsilon^{2-d} \to \infty$. The estimates from last week are useless in this limit, because they depend on $L^p(\Lambda_\varepsilon)$ norms of Y_t.

This is a problem of small scales. It hints to the fact that Y^ε is not converging to a function on $\mathbb{T}^d \approx [0,1]^d$, not even locally.

One way to deal with this problem and analyse what is going on in the equation (2) is to split all our functions in “blocks” which are nice.

Let us define $\mathbb{T}_\varepsilon^d = \Lambda_{\varepsilon,1} = (\varepsilon \mathbb{Z} \cap [-1/2, 1/2))^d$.

This here is accomplished via Littlewood–Paley decomposition, i.e. a nice partition of unity in Fourier space. We split every function $f: \mathbb{T}_\varepsilon^d \to \mathbb{R}$ in very nice pieces $(\Delta_i f)_{i \geqslant -1}$ as follows

$$f(x) = \sum_{i \geqslant -1} (\Delta_i f)(x)$$

where

$$\Delta_i f(x) := \sum_{k \in (\mathbb{Z} \cap [-\varepsilon^{-1}, \varepsilon^{-1}))^d} \rho(2^{-i} k) \hat{f}(k) e^{2\pi i k \cdot x} \qquad i \geqslant 0$$

and

$$\Delta_{-1} f(x) := \sum_{k \in (\mathbb{Z} \cap [-\varepsilon^{-1}, \varepsilon^{-1}))^d} \chi(k) \hat{f}(k) e^{2\pi i k \cdot x},$$

where $\rho \colon \mathbb{R}^d \to \mathbb{R}_{\geqslant 0}$ and is such that

$$\chi(k) + \sum_{i \geqslant 0} \rho(2^{-i} k) = 1$$

for a nice function $\chi \colon \mathbb{R}^d \to \mathbb{R}_{\geqslant 0}$ with support in a ball $\mathscr{B}$ of radius ≈ 1 around $k = 0 \in \mathbb{R}^d$ and ρ is supported on an annulus $\mathscr{A}$ of radius ≈ 1. All these functions are smooth (and some other properties we don't care about right now).

Therefore the Fourier transform of the LP block $\Delta_i f$ is supported on an annulus of size 2^i and that of $\Delta_{-1} f$ in a ball of radius 1.

Remark. For $\varepsilon > 0$ we have $\Delta_i f = 0$ if $2^i \gtrsim \varepsilon^{-1}$, so we sum over i up to $\approx \log_2 \varepsilon^{-1}$. Let us define N_ε to be this bound. So

$$f = \sum_{i=-1}^{N_\varepsilon} (\Delta_i f),$$

there is a technical subtelty here on how one handles the last last block but we will ignore it.

Now:

$$\mathbb{E}[(\Delta_i Y_t(x))^2] = \sum_{k \in (\mathbb{Z} \cap [-\varepsilon^{-1}, \varepsilon^{-1}))^d} \frac{\rho(2^{-i} k)^2}{(m^2 + \sum_i (2\varepsilon^{-1} \sin(\pi \varepsilon k_i))^2)}$$

$$\approx \sum_{k \in 2^i \mathscr{A} \subseteq (\mathbb{Z} \cap [-\varepsilon^{-1}, \varepsilon^{-1}))^d} \frac{\rho(2^{-i} k)^2}{(m^2 + |k|^2)} \approx \sum_{k \in 2^i \mathscr{A} \subseteq (\mathbb{Z} \cap [-\varepsilon^{-1}, \varepsilon^{-1}))^d} \frac{\rho(2^{-i} k)^2}{\underbrace{|k|^2}_{\approx (2^i)^2}} \lesssim (2^i)^{d-2}$$

Which says that $\Delta_i Y_t \approx (2^i)^{(d-2)/2}$ which is uniform in ε! (but of course not in i, and i can be large)

One can prove actually that $\Delta_i Y^\varepsilon$ converges to a nice C^∞ random function on $\mathbb{T}^d$ as $\varepsilon \to 0$.

This decomposition shift the problem of dealing with distribution to a problem of dealing with large sums.

Definition 1. *Let $\alpha \in \mathbb{R}$ and $p, q \in [1, +\infty]$. We say that $f \in B^\alpha_{p,q}$ (a Besov space) iff*

$$\|f\|_{B^\alpha_{p,q}} := \|i \geqslant -1 \mapsto 2^{i\alpha} \|\Delta_i f\|_{L^p}\|_{\ell^q} = \left[\sum_i (2^{i\alpha} \|\Delta_i f\|_{L^p})^q \right]^{1/q} < \infty.$$

These are Banach spaces.

In particular we will use $\mathscr{C}^\alpha = B^\alpha_{\infty,\infty}$ with norm $\|f\|_{\mathscr{C}^\alpha}$ such that

$$\|\Delta_i f\|_{L^\infty} \leqslant 2^{-\alpha i} \|f\|_{\mathscr{C}^\alpha}.$$

When $\alpha > 0$ there are spaces of regular functions, when $\alpha < 0$ these are just distributions (of course when $\varepsilon = 0$).

Moreover note that

$$\|f\|_{B_{2,2}^{\alpha}}^2 = \sum_{i\geqslant -1} 2^{2i\alpha}\|\Delta_i f\|_{L^2(\mathbb{T}_\varepsilon^d)}^2 = \varepsilon^d \sum_{x\in\mathbb{T}_\varepsilon^d}\sum_{i\geqslant -1} 2^{2i\alpha}|\Delta_i f(x)|^2 \approx \varepsilon^d \sum_{x\in\mathbb{T}_\varepsilon^d} |(1-\Delta_\varepsilon)^{\alpha/2} f|^2 =: \|f\|_{H^\alpha}^2.$$

From the estimate above on Y one can prove that almost surely for any $t\in\mathbb{R}$, $Y_t^\varepsilon \in \mathscr{C}^{(d-2)/2-\kappa}$ for any small $\kappa > 0$ uniformly in ε. One can also prove that as a function of t is continuous and actually uniformly in ε

$$Y^\varepsilon \in C(\mathbb{R}; \mathscr{C}^{(d-2)/2-\kappa})$$

almost surely.

In the following κ will always denote an arbitrary small positive quantity. (this is a small loss of regularity due to the fact that we want almost sure statements).

Note that when $d=2$ we have $Y^\varepsilon \in C(\mathbb{R}; \mathscr{C}^{-\kappa})$ and when $d=3$ $Y^\varepsilon \in C(\mathbb{R}; \mathscr{C}^{-1/2-\kappa})$.

Products in Besov spaces.

Take $f\in\mathscr{C}^\alpha$, $g\in\mathscr{C}^\beta$ then

$$fg = \sum_i \Delta_i f \sum_j \Delta_j g = \sum_{i,j} \Delta_i f \Delta_j g$$

to give a sense to this product one has to control the two (large) sums. The good way to do it is to split it in three pieces:

$$fg = \sum_{i,j} \Delta_i f \Delta_j g = \sum_{i<j-K} \Delta_i f \Delta_j g + \sum_{i>j+K} \Delta_i f \Delta_j g + \sum_{|i-j|\leq K} \Delta_i f \Delta_j g$$

$$=: (f \prec g) + (f \succ g) + (f \circ g)$$

and call them the paraproducts $(f\prec g)$, $(f\succ g) = (g \prec f)$ and the resonant term $(f\circ g)$.

Theorem 2. *The paraproducts are always well defined and*

$$\|f\prec g\|_{\mathscr{C}^\beta} \lesssim \|f\|_{\mathscr{C}^\alpha}\|g\|_{\mathscr{C}^\beta}, \qquad \alpha > 0,$$

$$\|f\prec g\|_{\mathscr{C}^{\alpha+\beta}} \lesssim \|f\|_{\mathscr{C}^\alpha}\|g\|_{\mathscr{C}^\beta}, \qquad \alpha < 0.$$

The resonant product is well-defined only if $\underline{\alpha+\beta>0}$ *and in this case*

$$\|f\circ g\|_{\mathscr{C}^{\alpha+\beta}} \lesssim \|f\|_{\mathscr{C}^\alpha}\|g\|_{\mathscr{C}^\beta}.$$

Therefore fg is well defined (and continuous) if $\alpha+\beta>0$ *and in this case*

$$\|fg\| \lesssim \|f\|_{\mathscr{C}^\alpha}\|g\|_{\mathscr{C}^\beta}.$$

We are allowed to multiply things only if regularity is ok, and the problem is in the resonant term.

Let's go back with these tools to our equation (2). In the r.h.s. we have

$$V'(Y+Z) = \lambda Y^3 + 3\lambda Y^2 Z + 3\lambda Y Z^2 + \lambda Z^3 + \beta Y + \beta Z.$$

Renormalization:

By the product theorem we see that Y^3 is problematic since the regularity $\alpha = (2-d)/2-\kappa$ of Y is negative. However Y is explicit, and we can do a probabilistic computation to prove that Y^3 converge as $\varepsilon \to 0$ to a well defined distribution provided it is *renormalized*.

Theorem 3. *There exists a constant c_ε such that the random field (renormalized square)*

$$\mathbb{Y}_t^{\varepsilon,2}(x) := (Y_t^\varepsilon(x))^2 - c_\varepsilon,$$

converges (in law) as $\varepsilon \to 0$ to a random field $\mathbb{Y}^2$ in $C(\mathbb{R}; \mathscr{C}^{2\alpha})$ with $\alpha = (2-d)/2-\kappa < 0$ (if $d \geqslant 2$). Similarly if $d = 2$ the renormalized cube

$$\mathbb{Y}_t^{\varepsilon,3}(x) := (Y_t^\varepsilon(x))^3 - 3c_\varepsilon Y_t^\varepsilon(x),$$

converges as $\varepsilon \to 0$ to a random field in $C(\mathbb{R}; \mathscr{C}^{3\alpha})$ while if $d = 3$ then convergence holds $C^{-\kappa}(\mathbb{R}; \mathscr{C}^{3\alpha})$ (where $C^{-\kappa}$ is a space of distributions in the time variable with negative regularity).

Moverover one can take

$$c_\varepsilon := \mathbb{E}[(Y_t^\varepsilon(x))^2] \approx \varepsilon^{(2-d)}.$$

With this choiche the renormalization corresponds to "Wick ordering".

Now we see that replacing

$$\beta = \beta_\varepsilon = \beta' - 3\lambda c_\varepsilon,$$

on has (with $\mathbb{Y}^1 = Y$)

$$V'(Y+Z) = \lambda \underbrace{(Y^3 - 3c_\varepsilon Y)}_{\mathbb{Y}^3} + 3\lambda \underbrace{(Y^2 - c_\varepsilon)}_{\mathbb{Y}^2} Z + 3\lambda Y Z^2 + \lambda Z^3 + \beta' Y + \beta' Z$$

$$= \lambda \mathbb{Y}^3 + 3\lambda \mathbb{Y}^2 Z + 3\lambda \mathbb{Y}^1 Z^2 + \lambda Z^3 + \beta' \mathbb{Y}^1 + \beta' Z.$$

Magic: one constant works for both problematic terms... (there are reasons for that, namely subcriticality of this model).

Next problems: the products

$$\underbrace{\mathbb{Y}^2}_{\mathscr{C}^{2\alpha}} Z, \qquad \underbrace{\mathbb{Y}^1}_{\mathscr{C}^{\alpha}} Z^2.$$

Let's try to get some estimates for Z: we test the equation (2) with Z and integrate in space $\mathbb{T}_\varepsilon^d$

$$\frac{1}{2}\frac{\partial}{\partial t}\int_{\mathbb{T}_\varepsilon^d} Z_t^2 + \int_{\mathbb{T}_\varepsilon^d} \left[|\nabla_\varepsilon Z_t|^2 + m^2 |Z_t|^2 + \frac{\lambda}{2}|Z_t|^4 \right]$$

$$= -\frac{1}{2}\int_{\mathbb{T}_\varepsilon^d} [\lambda \mathbb{Y}^3 Z + 3\lambda \mathbb{Y}^2 Z^2 + 3\lambda \mathbb{Y}^1 Z^3 + \beta' \mathbb{Y}^1 Z + \beta' Z^2].$$

The l.h.s tells me that I have control of the L^2, L^4 norm of Z but also of the H^1 norm of Z, this means we have some regularity for Z.

Note that $H^1 = B_{2,2}^1$. In the Besov scale we have Sobolev spaces. The theory of products and paraproducts extends naturally to Besov space with indexes p, q other than ∞, ∞.

When $d=2$ we have that $\mathbb{Y}^k \in \mathscr{C}^{-k\alpha}$ with $k=1,2,3$ and $\alpha=-\kappa$ a small negative quantity. Therefore all the products in the apriori r.h.s. are well defined assuming $Z \in H^1$ (the sums of regularities is positive!). For example one has estimates like (for some small δ and some large K)

$$\left| \int_{\mathbb{T}_\varepsilon^d} \mathbb{Y}^3 Z \right| \lesssim \| \mathbb{Y}^3 \|_{\mathscr{C}^{3\alpha}} \| Z \|_{B_{1,1}^{4\kappa}} \lesssim C_\delta \| \mathbb{Y}^3 \|_{\mathscr{C}^{3\alpha}}^K + \delta \| \nabla Z \|_{L^2}^2 + \delta \| Z \|_{L^2}^2,$$

$$\left| \int_{\mathbb{T}_\varepsilon^d} \mathbb{Y}^2 Z^2 \right| \lesssim \| \mathbb{Y}^2 \|_{\mathscr{C}^{2\alpha}} \| Z^2 \|_{B_{1,1}^{3\kappa}} \lesssim C_\delta \| \mathbb{Y}^2 \|_{\mathscr{C}^{2\alpha}}^K + \delta \| \nabla Z \|_{L^2}^2 + \delta \| Z \|_{L^4}^4,$$

$$\left| \int_{\mathbb{T}_\varepsilon^d} \mathbb{Y}^1 Z^3 \right| \lesssim \| \mathbb{Y}^1 \|_{\mathscr{C}^{\alpha}} \| Z^3 \|_{B_{1,1}^{2\kappa}} \lesssim C_\delta \| \mathbb{Y}^1 \|_{\mathscr{C}^{\alpha}}^K + \delta \| \nabla Z \|_{L^2}^2 + \delta \| Z \|_{L^4}^4,$$

So overall we can obtain

$$\frac{1}{2} \frac{\partial}{\partial t} \int_{\mathbb{T}_\varepsilon^d} Z_t^2 + (1-\delta) \int_{\mathbb{T}_\varepsilon^d} \left[|\nabla_\varepsilon Z_t|^2 + m^2 |Z_t|^2 + \frac{\lambda}{2} |Z_t|^4 \right] \leqslant Q(\mathbb{Y})$$

where

$$Q(\mathbb{Y}) := C \sum_{k=1,2,3} \| \mathbb{Y}_t^k \|_{\mathscr{C}^{k\alpha}}^K .$$

And it is a good apriori estimate in $d=2$ which are uniform in ε.

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