Lecture 8 | 22.2.2021 | 14:00–16:00 via Zoom

Web page: https://www.iam.uni-bonn.de/abteilung-gubinelli/sq-lectures-milan-ws2021
Recorded lectures: https://uni-bonn.sciebo.de/s/6mTx2gYCfCscfFm

An interesting recent talk of A. Jaffe "Is relativity compatible with quantum theory?"(December 2020) https://www.youtube.com/watch?v=RgQixyA2Gcs

It discusses the history and challenges in a mathematical theory of quantum fields.

Note that we define:

$$\int_{\mathbb{T}_\varepsilon^d} \mathrm{d}x := \varepsilon^d \sum_{x\in\mathbb{T}_\varepsilon^d} .$$

In $d=2$ we have obtained the apriori estimate (via PDE methods, no probability here)

$$\frac{1}{2}\frac{\partial}{\partial t}\int_{\mathbb{T}_\varepsilon^d} Z_t^2 + (1-\delta)\int_{\mathbb{T}_\varepsilon^d}\left[|\nabla_\varepsilon Z_t|^2 + m^2|Z_t|^2 + \frac{\lambda}{2}|Z_t|^4\right] \leqslant Q(\mathbb{Y}_t^\varepsilon),$$

where

$$Q(\mathbb{Y}_t^\varepsilon) := 1 + C\sum_{k=1,2,3}\|\mathbb{Y}_t^{\varepsilon,k}\|_{\mathscr{C}^{k\alpha}}^K$$

for some power K. This estimate holds for all paths of Y since note that $Y^\varepsilon \in C\left(\mathbb{R}\times\mathbb{R}^{\mathbb{T}_\varepsilon^d};\mathbb{R}\right)$ so it is clear that $Q(\mathbb{Y}_t^\varepsilon) < \infty$.

The real problem is, what happens when $\varepsilon \to 0$?

Remember that last week we constructed the stationary coupling $\mathbb{P}^\varepsilon$ such that under $\mathbb{P}^\varepsilon$ the processes Y and Z are stationary and

$$X = Y + Z$$

is also stationary and such that $X_t \sim \nu^\varepsilon$ (recall $M=1$ today).

Under this coupling this estimate implies that

$$\underbrace{\frac{1}{2}\frac{\partial}{\partial t}\mathbb{E}\int_{\mathbb{T}_\varepsilon^d} Z_t^2}_{=0 \text{ (by stationarity)}} + \mathbb{E}\int_{\mathbb{T}_\varepsilon^d}\left[|\nabla_\varepsilon Z_t|^2 + m^2|Z_t|^2 + \frac{\lambda}{2}|Z_t|^4\right] \lesssim \mathbb{E}Q(\mathbb{Y}_t^\varepsilon) = \mathbb{E}Q(\mathbb{Y}_0^\varepsilon),$$

but again, using some probability theory one can prove that

$$\sup_\varepsilon \mathbb{E}Q(\mathbb{Y}_0^\varepsilon) < \infty,$$

since again Y is well known and estimates are relatively easy. As a result one obtain unifom estimates of the form

$$\sup_\varepsilon \mathbb{E}\int_{\mathbb{T}_\varepsilon^d}\left[|\nabla_\varepsilon Z_0|^2 + m^2|Z_0|^2 + \frac{\lambda}{2}|Z_0|^4\right] < \infty.$$

This estimate is a key point because from that one can derive tightness of the family $(\nu^{\varepsilon})_{\varepsilon>0}$. Indeed let γ^{ε} the law of (Y_0, Z_0) under $\mathbb{P}^{\varepsilon}$, we have

$$\sup_{\varepsilon} \int (\|\psi\|^2_{\mathscr{C}^{-\alpha}} + \|\nabla\zeta\|^2_{L^2} + \|\zeta\|^2_{L^2} + \|\zeta\|^4_{L^4}) \gamma^{\varepsilon}(\mathrm{d}\psi \times \mathrm{d}\zeta)$$

$$= \sup_{\varepsilon} \mathbb{E}_{\mathbb{P}^{\varepsilon}}[\|Y_0\|^2_{\mathscr{C}^{-\alpha}} + \|\nabla Z_0\|^2_{L^2} + \|Z_0\|^2_{L^2} + \|Z_0\|^4_{L^4}] \lesssim \sup_{\varepsilon} \mathbb{E} Q(\mathbb{Y}_0^{\varepsilon}) < \infty,$$

note that $\|Y_0\|^2_{\mathscr{C}^{-\alpha}} \lesssim Q(\mathbb{Y}_0^{\varepsilon})$ if K large enough.

This gives tightness of $(\gamma^{\varepsilon})_{\varepsilon \geqslant 0}$ in $\mathscr{C}^{-2\alpha} \times (H^{1-\kappa} \cap L^4)$ (some loss of regularity to guarantee the required compactness). Projecting down to $(\nu^{\varepsilon})_{\varepsilon}$ (taking the sum of the two factors) one get tightness of $(\nu^{\varepsilon})_{\varepsilon}$ in $H^{-2\alpha} = B^{-2\alpha}_{2,2}$:

$$\int \|\varphi\|^2_{B^{-\alpha}_{2,2}} \nu^{\varepsilon}(\mathrm{d}\varphi) = \int \|\psi + \zeta\|^2_{B^{-\alpha}_{2,2}} \gamma^{\varepsilon}(\mathrm{d}\psi \times \mathrm{d}\zeta) \leqslant 2 \int (\|\psi\|^2_{B^{-\alpha}_{2,2}} + \|\zeta\|^2_{B^{-\alpha}_{2,2}}) \gamma^{\varepsilon}(\mathrm{d}\psi \times \mathrm{d}\zeta)$$

$$\leqslant 2 \int (\|\psi\|^2_{\mathscr{C}^{-\alpha}} + \|\zeta\|^2_{H^1}) \gamma^{\varepsilon}(\mathrm{d}\psi \times \mathrm{d}\zeta),$$

which is uniformly bounded in ε. So we can extract an accumulation point ν (a measure on $H^{-2\alpha}(\mathbb{T}^2)$.

Theorem 1. *Provided $d = 2$ and we take $\beta = -3\lambda c_{\varepsilon} + \beta'$ for some constant $\beta' \in \mathbb{R}$ and $c_{\varepsilon} = \mathbb{E}[Y_t^{\varepsilon}(x)^2]$ then the family $(\nu_{\varepsilon})_{\varepsilon}$ is tight in $H^{-2\alpha}(\mathbb{T}^2)$.*

What happens in $d = 3$. Let us go back to the apriori estimates: test the equation

$$\frac{\partial}{\partial t} Z_t = (\Delta_{\varepsilon} - m^2) Z_t - \frac{1}{2} V'(Y_t + Z_t) \quad (1)$$

where $V'(\varphi) = \lambda\varphi^3 + \beta\varphi$ (recall $\beta = -3\lambda c_{\varepsilon} + \beta'$) with Z and integrate in space $\mathbb{T}^3_{\varepsilon}$:

$$\frac{1}{2} \frac{\partial}{\partial t} \int_{\mathbb{T}^3_{\varepsilon}} Z_t^2 + \int_{\mathbb{T}^3_{\varepsilon}} \left[|\nabla_{\varepsilon} Z_t|^2 + m^2 |Z_t|^2 + \frac{\lambda}{2} |Z_t|^4 \right]$$

$$= -\frac{1}{2} \int_{\mathbb{T}^3_{\varepsilon}} [\lambda \mathbb{Y}^3 Z + 3\lambda \mathbb{Y}^2 Z^2 + 3\lambda \mathbb{Y}^1 Z^3 + \beta' \mathbb{Y}^1 Z + \beta' Z^2].$$

But now $\mathbb{Y}^2$ has regularity $-1 - 2\kappa$ and $\mathbb{Y}^3$ even worser than $-3/2 - 3\kappa$. For Z we can hope only for H^1 regularity from these estimates. *Big problem!!*

The term $\mathbb{Y}^1 Z^3, \mathbb{Y}^1 Z$ are still ok because $\mathbb{Y}^1$ has regularity $-1/2 - \kappa$.

We go back to the equation (1) and write it more explicitly

$$\frac{\partial}{\partial t} Z_t + (m^2 - \Delta_{\varepsilon}) Z_t = \underbrace{-\frac{1}{2} \lambda \mathbb{Y}^3}_{\mathscr{C}^{-3/2-3\kappa}} - \frac{3}{2} \lambda \mathbb{Y}^2 Z + \cdots$$

From the theory of parabolic equations one sees that Z cannot have better regularity that $2 + -3/2 - 3\kappa = 1/2 - 3\kappa > 0$ surely it cannot be H^1. Moreover in this case we even have a worser problem for the term $\mathbb{Y}^2 Z$ which is a prod. of something of reg. $-1 - 2\kappa$ and something of reg. $1/2 - 3\kappa$ which do not sum up to a positive quantity. The first step is to separated the problems in the product $\mathbb{Y}^2 Z$ via a decomposition, we write

$$\mathbb{Y}^2 Z = \mathbb{Y}^2 \succ Z + \mathbb{Y}^2 \preccurlyeq Z$$

where $\mathbb{Y}^2 \preccurlyeq Z = \mathbb{Y}^2 \prec Z + \mathbb{Y}^2 \circ Z$. By paraproduct estimates one has that $\mathbb{Y}^2 \succ Z$ has regularity of $\mathbb{Y}^2$ that is $-1-2\kappa$ and it is well-defined. The term containing the resonant product $\mathbb{Y}^2 \preccurlyeq Z$ is however not well defined.

Define a new stochastic object $\mathbb{Y}^{[3],\varepsilon}$ to be the solution of the equation

$$\frac{\partial}{\partial t}\mathbb{Y}_t^{[3],\varepsilon} + (m^2 - \Delta_\varepsilon)\, \mathbb{Y}_t^{[3],\varepsilon} = -\frac{1}{2}\lambda\, \mathbb{Y}_t^{3,\varepsilon},$$

(for example, take the stationary solution). Again this is a very explicit functional of the Gaussian process Y^ε and will be easy to analyze, in particular one can show that uniformly in ε it belongs to

$$\mathbb{Y}^{[3],\varepsilon} \in C(\mathbb{R}, \mathscr{C}^{1/2-3\kappa}(\mathbb{T}^3)),$$

in the sense that, for example,

$$\sup_\varepsilon \mathbb{E}\left[\sup_{t\in[0,T]} \|\mathbb{Y}_t^{[3],\varepsilon}\|_{\mathscr{C}^{1/2-3\kappa}(\mathbb{T}^3)}^K\right] < \infty,$$

for any K, T.

Now define $\mathbb{H}$ as the solution to

$$\frac{\partial}{\partial t}\mathbb{H}_t + (m^2 - \Delta_\varepsilon)\, \mathbb{H}_t = -\frac{1}{2}\lambda\, \mathbb{Y}_t^3 - \frac{3}{2}\lambda\, \mathbb{Y}_t^2 \succ \mathbb{H}_t,$$

this is a linear equation which can be easily solved and analysed and its solution $\mathbb{H}$ does not looks much different than $\mathbb{Y}^{[3],\varepsilon}$ and lives also in $C(\mathbb{R}, \mathscr{C}^{1/2-3\kappa}(\mathbb{T}^3))$.

Now define Φ as

$$Z =: \mathbb{H} + \Phi$$

which solves

$$\frac{\partial}{\partial t}\Phi_t + (m^2 - \Delta_\varepsilon)\, \Phi_t = -\frac{3}{2}\lambda\, \mathbb{Y}^2 \succ \Phi - \frac{3}{2}\lambda\, \mathbb{Y}^2 \preccurlyeq \Phi - \frac{3}{2}\lambda\, \mathbb{Y}^1 Z^2 - \frac{1}{2}\beta'(\mathbb{Y}^1 + Z) - \frac{\lambda}{2}Z^3. \tag{2}$$

This is the right equation to get apriori estimates for (almost since Φ cannot be expected to be in H^1 exaclty due to this equation). Let's test it with Φ to get

$$\frac{1}{2}\frac{\partial}{\partial t}\int_{\mathbb{T}_\varepsilon^3} \Phi_t^2 + \int_{\mathbb{T}_\varepsilon^3} \Phi_t (m^2 - \Delta_\varepsilon)\, \Phi_t + \frac{\lambda}{2}\int_{\mathbb{T}_\varepsilon^3} \Phi_t^4$$

$$= \int_{\mathbb{T}_\varepsilon^3} \Phi_t \left[-\frac{3}{2}\lambda\, \mathbb{Y}^2 \succ \Phi_t - \frac{3}{2}\lambda\, \mathbb{Y}^2 \preccurlyeq \Phi_t - \frac{3}{2}\lambda\, \mathbb{Y}_t^1 Z_t^2 - \frac{1}{2}\beta'(\mathbb{Y}^1 + Z_t)\right]$$

$$-\frac{\lambda}{2}\int_{\mathbb{T}_\varepsilon^3} \Phi_t ((\mathbb{H}_t + \Phi_t)^3 - \Phi_t^3)$$

We have now to cross fingers and check that all the terms in the r.h.s. can be controlled with the l.h.s.

The term

$$-\frac{\lambda}{2}\int_{\mathbb{T}_\varepsilon^3}\Phi_t((\mathbb{H}_t+\Phi_t)^3-\Phi_t^3),$$

is not scary at all since $\mathbb{H}$ is a nice function and it contains only powers less than 4 of Φ so it can be controlled via the $\int\Phi^4$ is the l.h.s. (like in the infinite vol estimates of last week). The term

$$-\frac{3}{2}\lambda\int_{\mathbb{T}_\varepsilon^3}\Phi_t\mathbb{Y}^1 Z_t^2=-\frac{3}{2}\lambda\int_{\mathbb{T}_\varepsilon^3}\Phi_t\mathbb{Y}^1\,(\mathbb{H}+\Phi_t)^2$$

is also fine since $\mathbb{Y}^1$ is only $-1/2-\kappa$ irregular and we have the H^1 norm of Φ and it is at most cubic in Φ^3. With some work one can get a nice estimate. Note however that this term will contain products

$$\mathbb{Y}^1\mathbb{H},\quad \mathbb{Y}^1\mathbb{H}^2,$$

which are not well defined because $\mathbb{H}$ is only of regularity $1/2-2\kappa$ so the reg. do not sum up to positive number. However these terms can be analysed with probabilistic estimates and shown to be well defined and not needing renormalization. We will assume in the following that they have uniform estimates as $\varepsilon\to 0$ in

$$\mathbb{Y}^1\mathbb{H},\mathbb{Y}^1\mathbb{H}^2\in C(\mathbb{R};\mathscr{C}^{-1/2-\kappa}(\mathbb{T}^3)).$$

I'm worried about the terms:

$$-\frac{3}{2}\lambda\int_{\mathbb{T}_\varepsilon^3}\Phi_t[\mathbb{Y}^2\succ\Phi_t+\mathbb{Y}^2\preccurlyeq\Phi_t]$$

since Φ is not regular enough for $\mathbb{Y}^2$. Here we use the following fact.

Lemma 2. *We have*

$$D(f,g,h):=\int_{\mathbb{T}_\varepsilon^3}f(g\succ h)-\int_{\mathbb{T}_\varepsilon^3}(g\circ f)h$$

is well defined and continuous when the sum of the regularities of f,g,h is positive. For example

$$|D(f,g,h)|\leqslant\|f\|_{H^\alpha}\|h\|_{H^\gamma}\|g\|_{\mathscr{C}^\beta}$$

whenever $\alpha+\beta+\gamma>0$.

Using this lemma we have

$$\int_{\mathbb{T}_\varepsilon^3}\Phi_t[\mathbb{Y}^2\succ\Phi_t+\mathbb{Y}^2\circ\Phi_t]=\int_{\mathbb{T}_\varepsilon^3}\Phi_t[2\mathbb{Y}^2\succ\Phi_t]+D(\Phi_t,\mathbb{Y}_t^2,\Phi_t)$$

We got rid of the resonant product but the term

$$\int_{\mathbb{T}_\varepsilon^3}\Phi_t[2\mathbb{Y}^2\succ\Phi_t]$$

is still dangerous.

Going back to the apriori estimate we focus on two terms

$$\cdots+\int_{\mathbb{T}_\varepsilon^3}\Phi_t(m^2-\Delta_\varepsilon)\,\Phi_t=-3\lambda\int_{\mathbb{T}_\varepsilon^3}\Phi_t[\mathbb{Y}^2\succ\Phi_t]+\cdots$$

and try to cancel the one in r.h.s. using that in the l.h.s. This is possible by defining

$$\Psi_t := \Phi_t + \frac{3\lambda}{2}(m^2 - \Delta_\varepsilon)^{-1}[\mathbb{Y}^2 \succ \Phi_t]$$

and substituting the estimate (i.e. we are completing the above square). One

$$\int_{\mathbb{T}_\varepsilon^3} \Phi_t (m^2 - \Delta_\varepsilon)\, \Phi_t$$

$$= \int_{\mathbb{T}_\varepsilon^3} \left[\Psi_t - \frac{3\lambda}{2}(m^2 - \Delta_\varepsilon)^{-1}[\mathbb{Y}^2 \succ \Phi_t]\right](m^2 - \Delta_\varepsilon)\left[\Psi_t - \frac{3\lambda}{2}(m^2 - \Delta_\varepsilon)^{-1}[\mathbb{Y}^2 \succ \Phi_t]\right]$$

$$= \int_{\mathbb{T}_\varepsilon^3} \Psi_t (m^2 - \Delta_\varepsilon)\, \Psi_t - 3\lambda \int_{\mathbb{T}_\varepsilon^3} \Psi_t [\mathbb{Y}^2 \succ \Phi_t] + \frac{9\lambda}{4}\int_{\mathbb{T}_\varepsilon^3} (\mathbb{Y}^2 \succ \Phi_t)(m^2 - \Delta_\varepsilon)^{-1}(\mathbb{Y}^2 \succ \Phi_t)$$

$$= \int_{\mathbb{T}_\varepsilon^3} \Psi_t (m^2 - \Delta_\varepsilon)\, \Psi_t - 3\lambda \int_{\mathbb{T}_\varepsilon^3} \Phi_t [\mathbb{Y}^2 \succ \Phi_t] - \frac{9\lambda}{4}\int_{\mathbb{T}_\varepsilon^3} (\mathbb{Y}^2 \succ \Phi_t)(m^2 - \Delta_\varepsilon)^{-1}(\mathbb{Y}^2 \succ \Phi_t)$$

So now our apriori estimate reads:

$$\frac{1}{2}\frac{\partial}{\partial t}\int_{\mathbb{T}_\varepsilon^3} \Phi_t^2 + \int_{\mathbb{T}_\varepsilon^3} \Psi_t (m^2 - \Delta_\varepsilon)\, \Psi_t + \frac{\lambda}{2}\int_{\mathbb{T}_\varepsilon^3} \Phi_t^4$$

$$= -\frac{9\lambda}{4}\int_{\mathbb{T}_\varepsilon^3} (\mathbb{Y}^2 \succ \Phi_t)(m^2 - \Delta_\varepsilon)^{-1}(\mathbb{Y}^2 \succ \Phi_t)$$

$$+ \int_{\mathbb{T}_\varepsilon^3} \Phi_t \left[-\frac{3}{2}\lambda \mathbb{Y}^2 \prec \Phi_t - \frac{3}{2}\lambda \mathbb{Y}_t^1 Z_t^2 - \frac{1}{2}\beta'(\mathbb{Y}^1 + Z_t)\right]$$

$$- \frac{\lambda}{2}\int_{\mathbb{T}_\varepsilon^3} \Phi_t ((\mathbb{H}_t + \Phi_t)^3 - \Phi_t^3)$$

With

$$\Phi_t = -\frac{3\lambda}{2}(m^2 - \Delta_\varepsilon)^{-1}[\mathbb{Y}^2 \succ \Phi_t] + \Psi_t,$$

with $\Psi_t \in H^1$ (this is what is called a “paracontrolled expansion” of the solution). This shows however that the regularity of Φ is only $1 - 2\kappa$ coming from the regularity of

$$(m^2 - \Delta_\varepsilon)^{-1}\mathbb{Y}^2.$$

On Thursday we will discuss the additional renormalizations needed to give a well defined limit to the quantities

$$\int_{\mathbb{T}_\varepsilon^3} (\mathbb{Y}^2 \succ \Phi_t)(m^2 - \Delta_\varepsilon)^{-1}(\mathbb{Y}^2 \succ \Phi_t)$$

and

$$\int_{\mathbb{T}_\varepsilon^3} \Phi_t \mathbb{Y}^1(\mathbb{H}^2 + \mathbb{H}Z_t) \approx \int_{\mathbb{T}_\varepsilon^3} \Phi_t \mathbb{Y}^1((\mathbb{Y}^{[3]})^2 + \mathbb{Y}^{[3]} \succ Z_t) + \cdots$$

which diverges as $\varepsilon \to 0$. This will be taken into account by a suitable choice of β'.

This will solve the $\varepsilon \to 0$ limit and then we will discuss how to put together the techniques to handle $\varepsilon \to 0$ and $M \to \infty$ limits.

To summarize what we have, so far we wrote the initial solution X with the following decomposition

$$X = \mathbb{Y}^1 + \overbrace{\mathbb{H} \underbrace{- \frac{3\lambda}{2}(m^2 - \Delta_\varepsilon)^{-1}[\mathbb{Y}^2 \succ \Phi_t] + \Psi_t}_{\Phi}}^{Z},$$

which is really an expansion of the solution in terms of increasing regularity:

term	reg
$\mathbb{Y}^1$	$-1/2-\kappa$
$\mathbb{H}$	$1/2-3\kappa$
$-\frac{3\lambda}{2}(m^2-\Delta_\varepsilon)^{-1}[\mathbb{Y}^2 \succ \Phi_t]$	$1-2\kappa$
Ψ	1

These lecture notes are produced using the computer program $\text{T}_{\text{E}}\text{X}_{\text{MACS}}$. If you want to know more go here `www.texmacs.org`.