Web page: https://www.iam.uni-bonn.de/abteilung-gubinelli/sq-lectures-milan-ws2021
Recorded lectures: https://uni-bonn.sciebo.de/s/6mTx2gYCfCscfFm

Today:

- Show the second renormalization (not present in $d=2$).
- Finish the discussion of apriori estimates, both in finite and then infinite volume, this will give tightness for the measure and existence of accumulation points.
- Give some properties of these accumulations points. (So far we do not have proofs of uniqueness within the SQ approach) Remark: uniqueness is expected when λ/m^2 small enough.

Recall our setting: we look at the equation in $d=3$,

$$\frac{\partial}{\partial t} Z_t = (\Delta_\varepsilon - m^2) Z_t - \frac{1}{2} V'(Y_t + Z_t) \tag{1}$$

in finite volume ($M=1$) and $\varepsilon > 0$, with

$$V'(\varphi) = \lambda \varphi^3 + \beta \varphi$$

(and as in two dimensions we take $\beta = -3\lambda c_\varepsilon + \beta'$).

$$\frac{\partial}{\partial t} Z_t = (\Delta_\varepsilon - m^2) Z_t - \frac{\lambda}{2}[\mathbb{Y}^3 + 3\mathbb{Y}^2 Z + 3\mathbb{Y} Z^2 + Z^3]$$

where

$$\mathbb{Y}^3 := Y^3 - 3c_\varepsilon Y, \qquad \mathbb{Y}^2 := Y^2 - c_\varepsilon,$$

We defined $\mathbb{H}$ to be

$$\frac{\partial}{\partial t} \mathbb{H}_t + (m^2 - \Delta_\varepsilon) \mathbb{H}_t = -\frac{\lambda}{2} \mathbb{Y}_t^3 - \frac{3\lambda}{2} \mathbb{Y}_t^2 \succ \mathbb{H}_t,$$

(solve this equation by a fixpoint) and defined $\Phi := Z - \mathbb{H}$ which satisfies

$$\frac{\partial}{\partial t} \Phi_t = (\Delta_\varepsilon - m^2) \Phi_t - \frac{\lambda}{2}[-3\lambda \mathbb{Y}^2 \succ \mathbb{H} + 3\mathbb{Y}^2 \succ Z + 3\mathbb{Y}^2 \circ Z + 3\mathbb{Y}^2 \prec Z + 3\mathbb{Y} Z^2 + Z^3]$$

$$= (\Delta_\varepsilon - m^2) \Phi_t - \frac{\lambda}{2}\left[3\mathbb{Y}^2 \succ \Phi + \underbrace{3\mathbb{Y}^2 \circ \Phi + 3\mathbb{Y}^2 \circ \mathbb{H}}_{\text{dangerous terms!!!}} + 3\mathbb{Y}^2 \prec Z + 3\mathbb{Y} Z^2 + Z^3\right].$$

Recall the various regularities (we use κ for an arbitrary small >0 which can be different from line to line)

term	reg
$\mathbb{Y}^1$	$-1/2-\kappa$
$\mathbb{Y}^2$	$-1-\kappa$
$\mathbb{Y}^3$	"$-3/2-\kappa$" (as space-time distribution)
$\mathbb{H}$	$1/2-\kappa$
$\mathbb{Y}^2 \succ \Phi$	$-1-\kappa$
Φ	$1-\kappa$

Then we tested with Φ to get

$$\frac{1}{2}\frac{\partial}{\partial t}\int_{\mathbb{T}_\varepsilon^3}\Phi_t^2+\int_{\mathbb{T}_\varepsilon^3}\Phi_t(m^2-\Delta_\varepsilon)\,\Phi_t+\frac{\lambda}{2}\int_{\mathbb{T}_\varepsilon^3}\Phi_t^4$$

$$=\int_{\mathbb{T}_\varepsilon^3}\Phi\left[-\frac{3}{2}\lambda\,\mathbb{Y}^2\succ\Phi-\frac{3}{2}\lambda\,\mathbb{Y}^2\circ\Phi\right]$$

$$+\int_{\mathbb{T}_\varepsilon^3}\Phi\left[-\frac{3}{2}\lambda\,\mathbb{Y}^2\circ\mathbb{H}-\frac{1}{2}\beta'(\mathbb{Y}^1+Z)\right]$$

$$-\frac{3}{2}\lambda\int_{\mathbb{T}_\varepsilon^3}\Phi(\mathbb{Y}^2\prec Z+\mathbb{Y}^1Z^2)-\frac{\lambda}{2}\int_{\mathbb{T}_\varepsilon^3}\Phi((\mathbb{H}+\Phi)^3-\Phi^3)$$

and we did a transformation to the combination (in which all the terms are "ill defined", i.e. I cannot hope to control them separately in the limit)

$$A:=\int_{\mathbb{T}_\varepsilon^3}\Phi\,(m^2-\Delta_\varepsilon)\,\Phi+\int_{\mathbb{T}_\varepsilon^3}\Phi\left[\frac{3}{2}\lambda\,\mathbb{Y}^2\succ\Phi+\frac{3}{2}\lambda\,\mathbb{Y}^2\circ\Phi\right]$$

we use a "commutator lemma" to replace $\int\Phi(\mathbb{Y}^2\circ\Phi)$ with $\int(\mathbb{Y}^2\succ\Phi)\Phi$ modulo nice error term:

$$A=\int_{\mathbb{T}_\varepsilon^3}\Phi\,(m^2-\Delta_\varepsilon)\,\Phi+\int_{\mathbb{T}_\varepsilon^3}\Phi[3\lambda\,\mathbb{Y}^2\succ\Phi]+\lambda D(\Phi,\mathbb{Y}^2,\Phi)$$

Then we defined Ψ so that

$$\Phi=-\frac{3\lambda}{2}(m^2-\Delta_\varepsilon)^{-1}[\mathbb{Y}^2\succ\Phi]+\Psi,$$

$$A=\int_{\mathbb{T}_\varepsilon^3}\Psi_t(m^2-\Delta_\varepsilon)\,\Psi_t+\frac{9\lambda}{4}\int_{\mathbb{T}_\varepsilon^3}(\mathbb{Y}^2\succ\Phi_t)\,(m^2-\Delta_\varepsilon)^{-1}(\mathbb{Y}^2\succ\Phi_t)+\lambda D(\Phi,\mathbb{Y}^2,\Phi)$$

At this point we have decomposed X as

$$X=\mathbb{Y}^1+\mathbb{H}-\frac{3\lambda}{2}(m^2-\Delta_\varepsilon)^{-1}[\mathbb{Y}^2\succ\Phi]+\Psi \tag{2}$$

where

$$\Phi=X-\mathbb{Y}^1-\mathbb{H}=-\frac{3\lambda}{2}(m^2-\Delta_\varepsilon)^{-1}[\mathbb{Y}^2\succ\Phi]+\Psi$$

with these different functions statisfying the apriori equation

$$\frac{1}{2}\frac{\partial}{\partial t}\int_{\mathbb{T}_\varepsilon^3}\Phi^2+\int_{\mathbb{T}_\varepsilon^3}\Psi\,(m^2-\Delta_\varepsilon)\,\Psi+\frac{\lambda}{2}\int_{\mathbb{T}_\varepsilon^3}\Phi^4$$

$$=\int_{\mathbb{T}_\varepsilon^3}\left[-\frac{9\lambda^2}{4}\int_{\mathbb{T}_\varepsilon^3}(\mathbb{Y}^2\succ\Phi_t)\,(m^2-\Delta_\varepsilon)^{-1}(\mathbb{Y}^2\succ\Phi_t)-\frac{3}{2}\lambda\,\Phi\,\mathbb{Y}^2\circ\mathbb{H}-\frac{1}{2}\beta'\Phi(\mathbb{Y}^1+Z)\right]$$

$$+\lambda D(\Phi,\mathbb{Y}^2,\Phi)-\frac{3}{2}\lambda\int_{\mathbb{T}_\varepsilon^3}\Phi(\mathbb{Y}^2\prec Z+\mathbb{Y}^1Z^2)-\frac{\lambda}{2}\int_{\mathbb{T}_\varepsilon^3}\Phi((\mathbb{H}+\Phi)^3-\Phi^3)$$

The good guys are on the l.h.s and the bad guys on the r.h.s., with the ugly guys in orange.

The terms in orange are still out of control, in particular they contain products which are not well defined (because the regularities do not sum up to positive).

Let us pause a moment and try to understand the meaning of the decomposition (2): this is the key point of these new approaches to singular SPDEs (i.e. regularity structures or paracontrolled distributions). The message is that we cannot just look at generic functions in a given vector space (like in classical PDE theory) but we need to specify the solution as an "expansion" in terms (explicit or implicit) of different character. In the paracontrolled approach this involves the regularity of the various terms

$$X = \underbrace{\mathbb{Y}^1}_{-1/2-\kappa} + \underbrace{\mathbb{H}}_{1/2-\kappa} \underbrace{- \frac{3\lambda}{2}(m^2 - \Delta_\varepsilon)^{-1}[\mathbb{Y}^2 \succ \Phi]}_{1-\kappa} + \underbrace{\Psi}_{H^1},$$

(actually Ψ is even better than H^1, if I remember correctly it has regularity $3/2$).

For example one could see from this that for the LP blocks one has

$$\begin{aligned}
&\Delta_i X && \sim (2^i)^{1/2+\kappa}, \\
&\Delta_i X - \Delta_i \mathbb{Y}^1 && \sim (2^i)^{-1/2-\kappa} \\
&\Delta_i X - \Delta_i \mathbb{Y}^1 - \Delta_i \mathbb{H} && \sim (2^i)^{-1-\kappa} \\
&\Delta_i X - \Delta_i \mathbb{Y}^1 - \Delta_i \mathbb{H} + \frac{3\lambda}{2}\Delta_i\{(m^2 - \Delta_\varepsilon)^{-1}[\mathbb{Y}^2 \succ \Phi]\} && \sim (2^i)^{-1}
\end{aligned}$$

which can be interpreted by saying that my solution lives in a very particular subspace of the space of Besov functions of regularity $-1/2$ (we could take for example $H^{-1/2-\kappa}$).

In particular the stochastic objects $\mathbb{Y}^1, \mathbb{H}, \mathbb{Y}^2$ do not have better regularity as those stated (i.e. they are almost surely not in $\mathscr{C}^{-1/2}, \mathscr{C}^{1/2}, \mathscr{C}^1$ (think about Hölder regularity of BM).

The second renormalization

We need to understand what is going on with the red term

$$\int_{\mathbb{T}_\varepsilon^3}\left[-\frac{9\lambda^2}{4}\int_{\mathbb{T}_\varepsilon^3}(\mathbb{Y}^2 \succ \Phi_t)(m^2 - \Delta_\varepsilon)^{-1}(\mathbb{Y}^2 \succ \Phi_t) - \frac{3}{2}\lambda\,\Phi\,\mathbb{Y}^2 \circ \mathbb{H} - \frac{1}{2}\beta'\Phi(\mathbb{Y}^1 + Z)\right]$$

which contains not-well defined producs.

Start with $\mathbb{Y}^2 \circ \mathbb{H}$: use the definition of $\mathbb{H}$ (where $\mathscr{L} = \partial_t + (m^2 - \Delta_\varepsilon)$)

$$\mathbb{H} = -\frac{\lambda}{2}\mathbb{Y}^{[3]} - \frac{3\lambda}{2}\mathscr{L}^{-1}(\mathbb{Y}^2 \succ \mathbb{H}),$$

recall also that $\mathbb{Y}^{[3]} = \mathscr{L}^{-1}\mathbb{Y}^3$ (with reg. $1/2 - \kappa$), and write it as

$$\mathbb{Y}^2 \circ \mathbb{H} = -\frac{\lambda}{2}\mathbb{Y}^2 \circ \mathbb{Y}^{[3]} - \frac{3\lambda}{2}\mathbb{Y}^2 \circ \mathscr{L}^{-1}(\mathbb{Y}^2 \succ \mathbb{H}).$$

For $\mathbb{Y}^2 \circ \mathbb{Y}^{[3]}$ we can show by probabilistic argumens involving Wick products (i.e. explicit formulas for polynomials of Gaussian) that one can define other polynomials $\mathbb{Y}^{2\circ[3]}$ and $\mathbb{Y}^{2\circ[2]}$

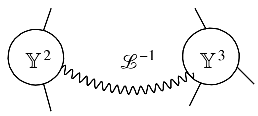
$$\mathbb{Y}^2 \circ \mathbb{Y}^{[3]} = \mathbb{Y}^2 \circ \mathscr{L}^{-1}\mathbb{Y}^3 = [\![Y^2]\!] \circ \mathscr{L}^{-1}[\![Y^3]\!] = \mathbb{Y}^{2\circ[3]} + 3d_\varepsilon \mathbb{Y}^1,$$

$$\mathbb{Y}^2 \circ \mathscr{L}^{-1}\mathbb{Y}^2 = \mathbb{Y}^{2\circ[2]} + d_\varepsilon$$

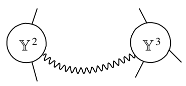
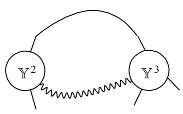
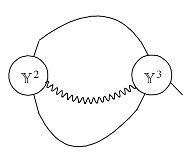
where d_ε is a constant which diverges logarithmically with ε. This is not much different from what we did in $d=2$ and in $d=3$ for the products Y^3, Y^2. The random field $\mathbb{Y}^{2\circ[3]}$ and $\mathbb{Y}^{2\circ[2]}$ converge as $\varepsilon \to 0$ to well defined random field such that

$$\boxed{\begin{aligned}\mathbb{Y}^{2\circ[3]} &:= \mathbb{Y}^2 \circ \mathbb{Y}^{[3]} - 3d_\varepsilon \mathbb{Y}^1 \in C(\mathbb{R}_+, \mathscr{C}^{1/2-\kappa})\\ \mathbb{Y}^{2\circ[2]} &:= \mathbb{Y}^2 \circ \mathscr{L}^{-1}\mathbb{Y}^2 - d_\varepsilon \in C(\mathbb{R}_+, \mathscr{C}^{-\kappa})\end{aligned}}$$

In terms of Feynman graphs one could write

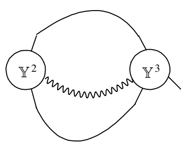
$\mathbb{Y}^2 \circ \mathbb{Y}^{[3]} =$ 

which can be decomposed in orthogonal terms

$=[\![$  $]\!] + 3\,2 [\![$  $]\!] + 3\,2^2 [\![$  $]\!]$

$$= [\![Y^2]\!] \diamond_0 \mathscr{L}^{-1}[\![Y^3]\!] + 3\,2[\![Y^2]\!] \diamond_1 \mathscr{L}^{-1}[\![Y^3]\!] + 3\,2^2[\![Y^2]\!] \diamond_2 \mathscr{L}^{-1}[\![Y^3]\!]$$

and one has that the last one is diverging while the other two are well defined

$[\![$  $]\!]$ $\approx \int_{\mathbb{T}_\varepsilon^3} \mathrm{d}x \underbrace{P(x-y)}_{\mathscr{L}^{-1}} \underbrace{G(x-y)^2}_{\text{two contraction lines}} Y(y) \approx \underbrace{\int_{\mathbb{T}_\varepsilon^3} \mathrm{d}x \frac{1}{|x|^3}}_{\propto d_\varepsilon} Y(y)$

since the correlation function

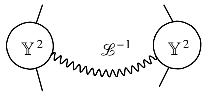
$$G(x-y) = \mathbb{E}[Y(x)Y(y)] \approx \sum_{k \in \mathbb{Z}^3 \cap [-\varepsilon^{-1}, \varepsilon^{-1}]^3} \frac{e^{ik(x-y)}}{k^2 + m^2} \approx \frac{1}{|x-y|}$$

and the kernel P of $\mathscr{L}^{-1}$ behaves in the same way

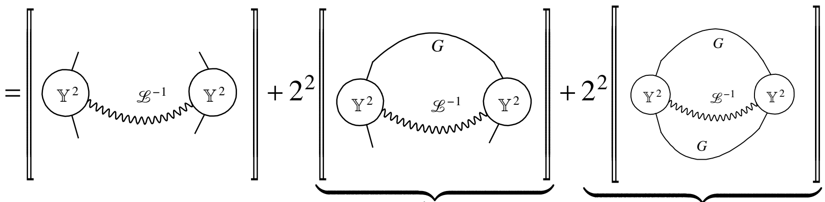
$$P(x-y) \approx |x-y|^{-1}.$$

For $\mathbb{Y}^2 \circ \mathscr{L}^{-1}\mathbb{Y}^2$ one can do the same:

$$\mathbb{Y}^2 \circ \mathbb{Y}^{[2]} =$$



$$=$$



$$\propto \int \frac{\mathrm{d}x}{|x|^2} < +\infty \qquad =: d_\varepsilon \propto \int \frac{\mathrm{d}x}{|x|^3} \approx \log \varepsilon^{-1}$$

$$= \mathbb{Y}^{2\circ[2]} + d_\varepsilon$$

where $\mathbb{Y}^{2\circ[2]}$ denotes the sum of the first two graphs.

Let us go back to $\mathbb{Y}^2 \circ \mathbb{H}$. The first step is to use commutator lemmas for paraproducs and resonant products:

> Commutator lemmas roughly say that one can usually write
>
> $$f \circ (g \succ h) \approx (f \circ g)h$$
>
> modulo "nice terms". Similar statements can be made when there are other nice linear operations in between, e.g.
>
> $$f \circ \mathscr{L}^{-1}(g \succ h) \approx (f \circ \mathscr{L}^{-1}g)h, \qquad f \circ (m^2 - \Delta)^{-1}(g \succ h) \approx (f \circ (m^2 - \Delta)^{-1}g)h$$

to show that

$$\mathbb{Y}^2 \circ \mathscr{L}^{-1}(\mathbb{Y}^2 \succ \mathbb{H}) = \underbrace{[\mathbb{Y}^2 \circ \mathscr{L}^{-1}(\mathbb{Y}^2)]}_{\text{ugly guy!}}\mathbb{H} + \underbrace{C(\mathbb{Y}^2, \mathbb{Y}^2, \mathbb{H})}_{\text{nice commutator}}$$

$$= (\mathbb{Y}^{2\circ[2]} + d_\varepsilon)\mathbb{H} + C(\mathbb{Y}^2, \mathbb{Y}^2, \mathbb{H})$$

Therefore we can handle the full term $\mathbb{Y}^2 \circ \mathbb{H}$ as

$$\mathbb{Y}^2 \circ \mathbb{H} = -\frac{\lambda}{2}\underbrace{\mathbb{Y}^2 \circ \mathbb{Y}^{[3]}}_{\text{uglyguy!}} - \frac{3\lambda}{2}\mathbb{Y}^2 \circ \mathscr{L}^{-1}(\mathbb{Y}^2 \succ \mathbb{H})$$

$$= -\frac{\lambda}{2}(\mathbb{Y}^{2\circ[3]} + 3d_\varepsilon \mathbb{Y}^1) - \frac{3\lambda}{2}(\mathbb{Y}^{2\circ[2]} + d_\varepsilon)\mathbb{H} - \frac{3\lambda}{2}C(\mathbb{Y}^2, \mathbb{Y}^2, \mathbb{H})$$

$$= -\frac{\lambda}{2}\mathbb{Y}^{2\circ[3]} - \frac{3\lambda}{2}\mathbb{Y}^{2\circ[2]}\mathbb{H} - \frac{3\lambda}{2}C(\mathbb{Y}^2, \mathbb{Y}^2, \mathbb{H}) - \frac{3\lambda}{2}d_\varepsilon(\mathbb{Y}^1 + \mathbb{H})$$

and we see precisely how $\mathbb{Y}^2 \circ \mathbb{H}$ diverges as $\varepsilon \to 0$, due to the presence of d_ε.

Now our task is to handle the other dangerous term (highlighted in red):

$$\int_{\mathbb{T}_\varepsilon^3} (\mathbb{Y}^2 \succ \Phi_t) \mathcal{Q}^{-1} (\mathbb{Y}^2 \succ \Phi_t)$$

with $\mathcal{Q} = (m^2 - \Delta_\varepsilon)$. We can decompose it with paraproducts and some commutator lemma as

$$B = \int_{\mathbb{T}_\varepsilon^3} (\mathbb{Y}^2 \succ \Phi_t) \mathcal{Q}^{-1} (\mathbb{Y}^2 \succ \Phi_t) = \int_{\mathbb{T}_\varepsilon^3} (\mathbb{Y}^2 \succ \Phi_t) \circ \mathcal{Q}^{-1} (\mathbb{Y}^2 \succ \Phi_t)$$

(only the resonant term counts in integrals)

$$B = \int_{\mathbb{T}_\varepsilon^3} (\mathbb{Y}^2 \circ \mathcal{Q}^{-1} \mathbb{Y}^2) \Phi^2 + \underbrace{C'(\mathbb{Y}^2, \mathbb{Y}^2, \Phi, \Phi)}_{\text{nice commutator}}$$

The same considerations as above apply to the explicit polynomial $\mathbb{Y}^2 \circ \mathcal{Q}^{-1} \mathbb{Y}^2$ and one defines

$$\mathbb{Y}^{2 \circ \{2\}} := \mathbb{Y}^2 \circ \mathcal{Q}^{-1} \mathbb{Y}^2 - d_\varepsilon$$

with the same constant as above. It is very similar to $\mathbb{Y}^2 \circ \mathscr{L}^{-1} \mathbb{Y}^2$, in particular the divergent part is the same! (very important). So the analysis of B gives

$$B = \int_{\mathbb{T}_\varepsilon^3} \mathbb{Y}^{2 \circ \{2\}} \Phi^2 + C'(\mathbb{Y}^2, \mathbb{Y}^2, \Phi, \Phi) + \int_{\mathbb{T}_\varepsilon^3} d_\varepsilon \Phi^2.$$

Putting all together we have

$$\int_{\mathbb{T}_\varepsilon^3} \left[-\frac{9\lambda^2}{4} \int_{\mathbb{T}_\varepsilon^3} (\mathbb{Y}^2 \succ \Phi_t)(m^2 - \Delta_\varepsilon)^{-1} (\mathbb{Y}^2 \succ \Phi_t) - \frac{3}{2} \lambda \Phi (\mathbb{Y}^2 \circ \mathbb{H}) - \frac{1}{2} \beta' \Phi (\mathbb{Y}^1 + Z) \right]$$

$$= -\frac{9\lambda^2}{4} \int_{\mathbb{T}_\varepsilon^3} \mathbb{Y}^{2 \circ \{2\}} \Phi^2 - \frac{9\lambda}{4} C'(\mathbb{Y}^2, \mathbb{Y}^2, \Phi, \Phi)$$

$$-\frac{3\lambda}{2} \int_{\mathbb{T}_\varepsilon^3} \Phi \left[-\frac{\lambda}{2} \mathbb{Y}^{2 \circ [3]} - \frac{3\lambda}{2} \mathbb{Y}^{2 \circ [2]} \mathbb{H} - \frac{3\lambda}{2} C(\mathbb{Y}^2, \mathbb{Y}^2, \mathbb{H}) \right]$$

$$-\frac{9\lambda^2}{4} \int_{\mathbb{T}_\varepsilon^3} \Phi [d_\varepsilon (\mathbb{Y}^1 + \mathbb{H} + \Phi)] \quad \text{[check signs!]}$$

$$-\frac{1}{2} \beta' \Phi (\mathbb{Y}^1 + Z),$$

and now the remarkable fact is that we can choose $\beta' = -9\lambda^2 d_\varepsilon / 2$ [check signs!] in order to cancel the divergences coming from d_ε. This means by choosing appropriately β we can remove all the divergences coming from ill-defined products of irregular Gaussian polynomials.

This is possible because this model is "superrenormalizable", or also called "subcritical", i.e. the linear part of the equation dominates the irregular terms in small scales, or said otherwise the non-linear irregular terms can be treated as a perturbation of the linear part.

We are at the point where in our apriori estimate we do not have any more ugly term, all the products are well defined with the available regularity and the only step remaining is to check that we can close the apriori estimates, i.e. estimate every term in the l.h.s. with the good terms in the r.h.s.
