Stochastic Analysis – Course note 1

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1 Stochastic Differential Equations

We introduce various notions of solutions to a stochastic differential equation (SDE) driven by a Brownian motion: weak solutions, strong solutions and martingale solutions. Uniqueness in law and pathwise uniqueness are two relevant concepts associated to these solutions. The Yamada–Watanabe theorem link these notions one to the other. Regularity of coefficients allow to prove existence of strong solutions.

1.1 Weak and strong solutions

Itô introduced SDE in the '40 with the aim of constructing diffusions, that is strong Markov processes with continuous paths and with generators which are second order differential operators. Stochastic analysis allows a pathwise approach to the construction of laws on path spaces and SDEs are the main tool for such constructions. SDE are also a natural approach to model physical systems which evolve in time and which are perturbed by "noise" (that is effect which we are not able to describe deterministically and for which we choose a probabilistic description). Recently, stochastic analysis has turned out to be a suitable tool to discuss mathematical finance (but Bachelier as early as the beginning of XX century introduced Brownian motion as a model of market prices). Stochastic analysis allows to easily remove the Markov hypothesis from the description of random processes (for example allowing memory in the coefficients) and more importantly allow to discuss the infinite dimensional situations more easily or in general certain Markov processes in very large state spaces (e.g. mean field models) in a relatively intuitive and direct fashion.

Main bibliographic references: [2, 3, 4].

We denote by $\mathcal{F}_\bullet = (\mathcal{F}_t)_{t \geqslant 0}$ a filtration. Let $C(\mathbb{R}_{\geqslant 0}; \mathbb{R}^N)$ be the space of continuous functions $\mathbb{R}_{\geqslant 0} \to \mathbb{R}^N$. On $C(\mathbb{R}_{\geqslant 0}; \mathbb{R}^N)$ we consider the canonical right-continuous filtration $(\mathcal{H}_t = \cap_{\varepsilon > 0} \sigma(Y_s; s \leqslant t + \varepsilon))_{t \geqslant 0}$ where Y is the canonical process on $C(\mathbb{R}_{\geqslant 0}; \mathbb{R}^N)$ defined by $Y_t(\omega) = \omega_t$. Moreover we let $\mathcal{H}_\infty = \vee_{t \geqslant 0} \mathcal{H}_t$ and note that $\mathcal{H}_\infty$ coincides with the Borel σ-field $\mathcal{B}(C(\mathbb{R}_{\geqslant 0}; \mathbb{R}^N))$. The predictable σ-field $\mathcal{P}_\bullet$ is the σ–field generated by left-continuous $(\mathcal{H}_t)_t$–adapted processes on $C(\mathbb{R}_{\geqslant 0}; \mathbb{R}^N)$.

Fix $D, M \geqslant 1$ and let $\sigma: \mathbb{R}_{\geqslant 0} \times C(\mathbb{R}_{\geqslant 0}; \mathbb{R}^D) \to \mathbb{R}^{D \times M}$ and $b: \mathbb{R}_{\geqslant 0} \times C(\mathbb{R}_{\geqslant 0}; \mathbb{R}^D) \to \mathbb{R}^D$ be two predictable processes. We will denote $\sigma = (\sigma^{i,j})_{i=1,\ldots,D, j=1,\ldots,M}$ and $b = (b^i)_{i=1,\ldots,D}$ the respective components. We want to study the multidimensional *stochastic differential equation* (SDE)

$$\mathrm{d}X_t^i = b_t^i(X)\mathrm{d}t + \sum_{j=1}^{M} \sigma_t^{i,j}(X)\mathrm{d}B_t^j, \qquad X_0 = x \in \mathbb{R}^D \qquad i = 1, \ldots, M, t \geqslant 0 \tag{1}$$

Definition 1. *A **(weak) solution** to the SDE (1) is given by pair of stochastic processes (X, B) on a filtered probability space $(\Omega, \mathcal{F}, \mathcal{F}_\bullet, \mathbb{P})$ such that*

i. B is a M–dimensional $(\mathcal{F}_\bullet, \mathbb{P})$–Brownian motion;

ii. for any $t \geqslant 0$,

$$\sum_i \int_0^t |b_s^i(X)| \, \mathrm{d}s + \sum_{i,j} \int_0^t (\sigma_s^{i,j}(X))^2 \, \mathrm{d}s < +\infty, \qquad \mathbb{P}\text{–}a.s.;$$

iii. for any $t \geqslant 0$, $i = 1, \ldots, d$,

$$X_t^i = x + \int_0^t b_s^i(X) \mathrm{d}t + \sum_{j=1}^M \int_0^t \sigma_s^{i,j}(X) \mathrm{d}B_s^j, \qquad \mathbb{P}\text{–}a.s.. \tag{2}$$

In what follows we will always use vector notations and rewrite (2) as

$$X_t = x + \int_0^t b_s(X) \mathrm{d}t + \int_0^t \sigma_s(X) \mathrm{d}B_s$$

where $\sigma_s(X)$ is understood as a linear map $\mathbb{R}^M \to \mathbb{R}^D$ acting on $\mathrm{d}B_s$. Unless otherwise stated on $\mathbb{R}^N$ we will consider the Euclidean norm (which is equivalent to the Hilbert–Schmidt norm for linear maps $\mathbb{R}^M \to \mathbb{R}^D$ seen as elements of $\mathbb{R}^{M \times D}$)

Moreover, note that the initial condition is part of the definition of solution. In general the initial condition can be random.

Definition 2. *A (weak) solution (X, B) is a **strong solution** if X adapted to the filtration $\mathcal{F}_\bullet^{B,\mathbb{P}}$, i.e. the filtration generated by B and completed according to $\mathbb{P}$.*

Definition 3. *We have **uniqueness in law** for the SDE (1) if for any pair of weak solutions (X, B) and $(\tilde{X}, \tilde{B})$ we have $\mathrm{Law}(X) = \mathrm{Law}(\tilde{X})$ on $C(\mathbb{R}_{\geqslant 0}; \mathbb{R}^D)$.*

Definition 4. *The SDE (1) has **pathwise uniqueness** if any pair of weak solutions (X, B) and $(\tilde{X}, B)$ (defined on the same probability space) are indistinguishable. (If the initial condition is random we require also that $X_0 = \tilde{X}_0$ a.s.)*

The following statements clarify the relations between these definitions.

Theorem 5. *If (X, B) is a strong solution of (1) then*

i. there exists a measurable map $\Phi: C(\mathbb{R}_{\geqslant 0}; \mathbb{R}^D) \to C(\mathbb{R}_{\geqslant 0}; \mathbb{R}^D)$ (endowed with the Borel σ-field) such that $X = \Phi(B)$ $\mathbb{P}$–a.s. ;

ii. for any $(\tilde{\mathcal{F}}_\bullet, \tilde{\mathbb{P}})$–Brownian motion $\tilde{B}$ on $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathcal{F}}_\bullet, \tilde{\mathbb{P}})$ the pair $(\Phi(\tilde{B}), \tilde{B})$ is a strong solution to (1).

Theorem 6. (Yamada–Watanabe) *If pathwise uniqueness holds, then*

i. *uniqueness in law holds.*

ii. *here exists a measurable map* $\Phi: C(\mathbb{R}_{\geqslant 0}; \mathbb{R}^D) \rightarrow C(\mathbb{R}_{\geqslant 0}; \mathbb{R}^D)$ *(endowed with the Borel* σ*-field) such that any weak solution* (X, B) *satisfies* $X = \Phi(B)$ $\mathbb{P} - a.s.$ *(therefore any weak solution is a strong solution and all the strong solutions coincide up to indistinguishability) .*

For a proof see [5] or [3, Chap. IX,Thm. 1.7].

Uniqueness in law could be formulated with respect to the *joint law* of the pair (X, B). A result of Cherny shows that the two concepts are equivalent and that together with existence of strong solutions they imply pathwise uniqueness.

Proposition 7. *If uniqueness in law holds then*

i. *for any pair of solutions* (X, B) *and* $(\tilde{X}, \tilde{B})$ *we have* $\mathrm{Law}(X, B) = \mathrm{Law}(\tilde{X}, \tilde{B})$.

ii. *existence of a strong solution implies pathwise uniqueness.*

For a proof see [1].

So, overall, the situation is the following:

a) It may happen that there are no solution in any probability space;

b) It may happen that there are (maybe multiple) solutions on a probability space but none on another;

c) if there exists a strong solution on a probability space then it is possible to construct solutions on any other probability space (carrying a Brownian motion). However there may be multiple solutions.

d) If pathwise uniqueness holds and there exists a solution on some probability space, then on any other probability space (carrying a Brownian motion) there exists only one solution and it is strong (Yamada–Watanabe).

e) The same ideal situation of point d) is reached if uniqueness in law holds and there exists a strong solution.

2 Martingale problems

Let $a: \mathbb{R}_{\geqslant 0} \times C(\mathbb{R}_{\geqslant 0}; \mathbb{R}^D) \rightarrow \mathbb{R}^{D \times D}$ and $b: \mathbb{R}_{\geqslant 0} \times C(\mathbb{R}_{\geqslant 0}; \mathbb{R}^D) \rightarrow \mathbb{R}^D$ be predictable functionals such that the matrix $a_t(x)$ is positive definite for all $t \geqslant 0$ and $x \in C(\mathbb{R}_{\geqslant 0}; \mathbb{R}^D)$.

Definition 8. *The process* X *is the solution to the martingale problem* $\mathrm{MP}(x_0, b, a)$ *if*

i. *for any* $t \geqslant 0$,

$$X_0 = x_0, \qquad \sum_{i=1}^{D} \int_0^t |b_s^i(X)| \, \mathrm{d}s + \sum_i \int_0^t a_s^{i,i}(X) \, \mathrm{d}s < +\infty, \qquad \mathbb{P}\text{–}a.s.; \tag{3}$$

ii. *the processes*

$$M_t^i = X_t^i - X_0^i - \int_0^t b_s^i(X)\mathrm{d}s, \qquad t \geqslant 0, i = 1, \ldots, D$$

$$N_t^{i,j} = M_t^i M_t^j - \int_0^t a_s^{i,j}(X)\mathrm{d}s, \qquad t \geqslant 0, i, j = 1, \ldots, D$$

are $(\mathcal{F}_\bullet^X, \mathbb{P})$ *continous local martingales.*

Proposition 9. *Let* $a = \sigma\sigma^T$ *and* $X_0 = x_0 \in \mathbb{R}^D$.

i. *If* (X, B) *is a solution to the SDE (1) then* X *is the solution to the martingale problem* $\mathrm{MP}(x_0, b, a)$*;*

ii. *If* X *is a solution to the martingale problem* $\mathrm{MP}(x_0, b, a)$ *then there exists a filtered probability space* $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathcal{F}}_\bullet, \tilde{\mathbb{P}})$ *carrying a solution* $(\tilde{X}, \tilde{B})$ *of (1) such that* $\mathrm{Law}(\tilde{X}) = \mathrm{Law}(X)$.

For a proof see [2].

Martingale problems allow to consider only the process X instead of the pair (X, B) needed in the definition of solutions to the SDE. Nonetheless, as the previous proposition shows, a suitable Brownian motion can always be added to a solution (at the price of changing the probability space), in order to satisfy the SDE. Martingale problems thus come handy in some questions related to transformations of weak solutions where the exact expression of the Brownian motion is not very important.

Lemma 10. *An equivalent formulation to the martingale problem* $\mathrm{MP}(x_0, b, a)$ *is to require that* X *satisfies (3) and that for all* $f \in C^2(\mathbb{R}^D)$ *the process* $M^{[f]}$ *given by*

$$M_t^{[f]} = f(X_t) - f(X_0) - \int_0^t \mathcal{L}f(X_s)\mathrm{d}s$$

is a $(\mathcal{F}_\bullet^X, \mathbb{P})$ *continous local martingale. Here* $\mathcal{L}$ *is the generator of the martingale problem given by*

$$\mathcal{L}f(t, X) = b_t(X).\nabla f(X_t) + \frac{1}{2}\mathrm{Trace}\,[a_t(X)\nabla^2 f(X_t)] \qquad f \in C^2(\mathbb{R}^D).$$

3 Sufficient conditions for existence and uniqueness

The more general result on uniqueness is the one for Lipschitz path–dependent coefficients:

Theorem 11. **(Itô)** *Assume that there exists a constant* C *such that for all* $t \geqslant 0$ *and* x, $y \in C(\mathbb{R}_{\geqslant 0}; \mathbb{R}^D)$ *we have*

$$|b_t(x) - b_t(y)| + |\sigma_t(x) - \sigma_t(y)| \leqslant C\|x - y\|_{\infty,[0,t]},$$

$$|b_t(x)| + |\sigma_t(x)| \leqslant C(1 + \|x\|_{\infty,[0,t]}).$$

Then strong existence and pathwise uniqueness holds.

For a proof see [3].

In what follows we will restrict our considerations to coefficients which depends only on the present, namely

$$b_t(x) = b(t, x_t) \quad \text{and} \quad \sigma_t(x) = \sigma(t, x_t), \qquad t \geqslant 0, x \in C(\mathbb{R}_{\geqslant 0}; \mathbb{R}^D).$$

where $b: \mathbb{R}_{\geqslant 0} \times \mathbb{R}^D \to \mathbb{R}^D$ and $\sigma: \mathbb{R}_{\geqslant 0} \times \mathbb{R}^D \to \mathbb{R}^{D \times M}$ are measurable functions.

In one dimension we can relax the Lipschitz assumption on the diffusion coefficient up to a condition of Hölder regularity of $1/2$:

Theorem 12. (Yamada–Watanabe) *Assume $D = 1$, $b_t(x) = b(t, x_t)$ and $\sigma_t(x) = \sigma(t, x_t)$ and that there exists $C, \gamma > 0$ and an increasing function $h: \mathbb{R}_{\geqslant 0} \to \mathbb{R}_{\geqslant 0}$ such that*

$$\int_0^\gamma \frac{\mathrm{d}s}{h^2(s)} = +\infty$$

and

$$|b(t, x) - b(t, y)| \leqslant C|x - y|, \quad |\sigma(t, x) - \sigma(t, y)| \leqslant h(|x - y|), \qquad t \geqslant 0, \quad x, y \in \mathbb{R};$$

then pathwise uniqueness holds for (1).

For a proof see [4, Ch. V, Th. 40.1]. Weak existence can be established for continuous coefficients :

Theorem 13. (Skorokhod) *Assume that b, σ are continuous and bounded. Then there exists a weak solutions of (1).*

For a proof see [4, Ch. V, Th. 23.5].

Other results on existence/uniqueness are available, see [2].

Theorem 14. (Stroock–Varadhan) *Let $D = M$ and assume that b is measurable and bounded and that σ is continuous, bounded and such that for all $t \geqslant 0$ and $x \in \mathbb{R}^D$ there exists a constant $\varepsilon(t, x) > 0$ such that*

$$|\sigma(t, x)v| \geqslant \varepsilon(t, x)|v|, \qquad v \in \mathbb{R}^D.$$

Then there exists a weak solution and uniqueness in law holds.

Bibliography

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