Stochastic Analysis – Course note 2

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1 Girsanov transformation

Girsanov (Maruyama, Cameron and Martin, Ramer) transformation describes the relation between modifying semimartingales via a finite variation process and absolutely continuous changes of probability measures. In particular it describes the quasi–invariance of Wiener measure wrt. adapted translations with values in the Cameron–Martin space. We first analyse general change of measures in presence of a filtration and a general form of Girsanov theorem. Then we prove some applications to SDEs and to the computation of conditional laws on path space. The main reference is [5, IV.38 and IV.39].

1.1 Change of measure on a filtered probability space

Let $\mathbb{P},\mathbb{Q}$ two probabilities on a filtered measure space $(\Omega,\mathcal{F},\mathcal{F}_\bullet)$. We will assume that $\mathcal{F}_\bullet$ is right continuous. If $\mathbb{Q}\ll\mathbb{P}$ on $\mathcal{F}_\infty=\vee_{t\geqslant 0}\mathcal{F}_t$ let

$$Z_\infty := \frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}}\Big|_{\mathcal{F}_\infty}$$

and note that for all $t>0$, $\mathbb{Q}\ll\mathbb{P}$ on $\mathcal{F}_t$ with

$$\mathbb{E}_{\mathbb{P}}[Z_\infty|\mathcal{F}_t] = \frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}}\Big|_{\mathcal{F}_t} =: Z_t \tag{1}$$

so the process $(Z_t)_{t\geqslant 0}$ is a uniformly integrable martingale closed by Z_∞. In what follows we will make the main assumption

$$(Z_t)_{t\geqslant 0} \text{ is a } \mathbb{P}\text{–a.s. continuous martingale.} \tag{2}$$

Lemma 1.

i. If T is a stopping time then

$$\frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}}\Big|_{\mathcal{F}_T} = Z_T; \tag{3}$$

ii. If $\mathbb{Q}\sim\mathbb{P}$ (i.e. $\mathbb{Q}\ll\mathbb{P}$ and $\mathbb{P}\ll\mathbb{Q}$) then $\mathbb{P}(Z_t>0$ for all $t\geqslant 0)=1$.

iii. If $\mathbb{Q}\sim\mathbb{P}$ then for any $t\geqslant s$ and $X\,\hat{\in}\,\mathcal{F}_t$ and $X\geqslant 0$ we have (Bayes formula)

$$\mathbb{E}_{\mathbb{Q}}[X|\mathcal{F}_s] = \frac{\mathbb{E}_{\mathbb{P}}[XZ_t|\mathcal{F}_s]}{Z_s} \qquad \mathbb{Q} \text{ and } \mathbb{P} \text{ a.s.} \tag{4}$$

iv. If $\mathbb{Q}\sim\mathbb{P}$ *then*

$$\begin{array}{lcl} M \text{ is a } \mathbb{Q}\text{–martingale} & \Longleftrightarrow & MZ \text{ is a } \mathbb{P}\text{–martingale} \\ M \text{ is a local } \mathbb{Q}\text{–martingale} & \Longleftrightarrow & MZ \text{ is a local } \mathbb{P}\text{–martingale} \end{array} \tag{5}$$

Proof. (i) If $A\in\mathcal{F}_T$, by optional sampling :

$$\mathbb{E}_{\mathbb{P}}[\mathbb{I}_A]=\mathbb{E}_{\mathbb{Q}}[Z_\infty\mathbb{I}_A]=\mathbb{E}_{\mathbb{Q}}[\mathbb{E}_{\mathbb{Q}}[Z_\infty|\mathcal{F}_T]\mathbb{I}_A]=\mathbb{E}_{\mathbb{Q}}[Z_T\mathbb{I}_A],$$

since Z_T is $\mathcal{F}_T$ measurable we proved the claim. (ii) Let $T=\inf\{t\geqslant 0\colon Z_t=0\}$, by continuity $Z_T=0$ on the event $A=\{T<\infty\}\in\mathcal{F}_T$. Then

$$\mathbb{Q}(A)=\mathbb{E}_{\mathbb{Q}}[\mathbb{I}_A]=\mathbb{E}_{\mathbb{P}}[Z_\infty\mathbb{I}_A]=\mathbb{E}_{\mathbb{P}}[Z_T\mathbb{I}_A]=0.$$

Since $\mathbb{P}\ll\mathbb{Q}$ we have $\mathbb{P}(A)=0$. (iii) For any $t\geqslant s\geqslant 0$ and $A\in\mathcal{F}_s$ we have

$$\mathbb{E}_{\mathbb{P}}[X\,\mathbb{I}_A]=\mathbb{E}_{\mathbb{Q}}[Z_\infty X\,\mathbb{I}_A]=\mathbb{E}_{\mathbb{Q}}[Z_tX\,\mathbb{I}_A]=\mathbb{E}_{\mathbb{Q}}[\mathbb{E}_{\mathbb{Q}}[Z_tX\,|\mathcal{F}_s]\,\mathbb{I}_A]=\mathbb{E}_{\mathbb{P}}\left[\frac{\mathbb{E}_{\mathbb{Q}}[Z_tX\,|\mathcal{F}_s]}{Z_s}\,\mathbb{I}_A\right]$$

and the claim follows. Let us prove (iv). The first equivalence is clear from Bayes formula (4). Let us consider the case of local martingales. Let T be a stopping time such that $(MZ)^T$ is a $\mathbb{Q}$ martingale. Then for all $t\geqslant s\geqslant 0$ and $A\in\mathcal{F}_s$ we have

$$\mathbb{E}_{\mathbb{P}}[M_t^T\mathbb{I}_A]=\mathbb{E}_{\mathbb{P}}[M_t^T\mathbb{I}_A\mathbb{I}_{T\leqslant s}]+\mathbb{E}_{\mathbb{P}}[M_t^T\mathbb{I}_A\mathbb{I}_{T>s}].$$

Now $\mathbb{E}_{\mathbb{P}}[M_t^T\mathbb{I}_A\mathbb{I}_{T\leqslant s}]=\mathbb{E}_{\mathbb{P}}[M_s^T\mathbb{I}_A\mathbb{I}_{T\leqslant s}]$ since here $T\leqslant s\leqslant t$. On the other hand $\mathbb{I}_A\mathbb{I}_{T>s}$ is $\mathcal{F}_{T\wedge s}\subseteq\mathcal{F}_{T\wedge t}$ measurable, so by (3) we have

$$\mathbb{E}_{\mathbb{P}}[M_t^T\mathbb{I}_A\mathbb{I}_{T>s}]=\mathbb{E}_{\mathbb{Q}}[Z_{T\wedge t}\,M_{T\wedge t}\mathbb{I}_A\mathbb{I}_{T>s}]=\mathbb{E}_{\mathbb{Q}}[(Z\,M)_s^T\,\mathbb{I}_A\mathbb{I}_{T>s}]$$

since $\mathbb{I}_A\mathbb{I}_{T>s}$ is $\mathcal{F}_s$–measurable and $(MZ)^T$ a $\mathbb{Q}$–martingale. But now

$$\mathbb{E}_{\mathbb{Q}}[(Z\,M)_s^T\mathbb{I}_A\mathbb{I}_{T>s}]=\mathbb{E}_{\mathbb{Q}}[Z_sM_s\mathbb{I}_A\mathbb{I}_{T>s}]=\mathbb{E}_{\mathbb{P}}[M_s\mathbb{I}_A\mathbb{I}_{T>s}]=\mathbb{E}_{\mathbb{P}}[M_s^T\mathbb{I}_A\mathbb{I}_{T>s}]$$

since $M_s\,\mathbb{I}_A\mathbb{I}_{T>s}$ is also $\mathcal{F}_s$–measurable. Then

$$\mathbb{E}_{\mathbb{P}}[M_t^T\mathbb{I}_A]=\mathbb{E}_{\mathbb{P}}[M_s^T\mathbb{I}_A\mathbb{I}_{T\leqslant s}]+\mathbb{E}_{\mathbb{P}}[M_s^T\mathbb{I}_A\mathbb{I}_{T>s}]=\mathbb{E}_{\mathbb{P}}[M_s^T\mathbb{I}_A]$$

which proves that M^T is a $\mathbb{P}$–martingale. The reverse implication can be proven similarly. Taking a localizing sequence of stopping times $(T_n)_n$ allows to conclude. □

Lemma 2. ***(Stochastic exponential, Doleans)*** *Let* $(Z_t)_{t\geqslant 0}$ *be a continuous local* $\mathbb{P}$*–martingale such that* $Z_t>0$ *for all* $t\geqslant 0$ $\mathbb{P}$*–almost surely then there exists a unique continuous local* $\mathbb{P}$*–martingale* $(L_t)_{t\geqslant 0}$ *such that*

$$Z_t=\exp\!\left(L_t-\frac{[L]_t}{2}\right)=:\mathcal{E}(L)_t \tag{6}$$

and given by the formula

$$L_t = \log Z_0 + \int_0^t \frac{\mathrm{d}Z_s}{Z_s}. \tag{7}$$

Proof. (Uniqueness) Assume $\tilde{L}$ is another such martingale. Then the local martingale $D = L - \tilde{L}$ satisfies $D_0 = 0$ and $2D_t = [L]_t - [\tilde{L}]_t$. Which implies that $D_t = 0$ for all $t \geqslant 0$ since D is a local martingale of bounded variation starting at 0. (Existence) Let L be defined by (7), applying Itô formula to $t \mapsto \log(Z_t)$ (possible since $Z_t > 0$ for all $t \geqslant 0$ and $z \mapsto \log(z)$ is C^2 away from $z = 0$) we get the claim, since

$$\mathrm{dlog}(Z_t) = \frac{\mathrm{d}Z_t}{Z_t} - \frac{1}{2}\frac{[Z]_t}{Z_t^2} = L_t - \frac{1}{2}[L]_t$$

given that $\mathrm{d}Z_t / Z_t = \mathrm{d}L_t$ and $\mathrm{d}[L]_t = \mathrm{d}[Z]_t / Z_t^2$ from eq. (7). □

Remark 3. Uniqueness of L can be also checked via Itô formula. Let $D_t = \exp(-L_t + [L]_t / 2)$ where L is defined by (7). Applying Itô formula to $D\,Z$ one has

$$\mathrm{d}(D\,Z)_t = Z_t \mathrm{d}D_t + D_t \mathrm{d}Z_t + \mathrm{d}[D, Z]_t = Z_t D_t(-\mathrm{d}L_t + \mathrm{d}[L]_t / 2) + \frac{Z_t D_t}{2}\mathrm{d}[L]_t + D_t \mathrm{d}Z_t - D_t \mathrm{d}[L, Z]_t,$$

using $\mathrm{d}L = \mathrm{d}Z / Z$ and $\mathrm{d}D = -D\mathrm{d}L$ we get $\mathrm{d}(D\,Z)_t = 0$ so $Z_t = (Z_0 D_0) D_t^{-1} = \exp(-L_t + [L]_t / 2)$ since $D_0 Z_0 = 1$.

Theorem 4. (Girsanov) *Let $\mathbb{P} \sim \mathbb{Q}$ and let Z be martingale defined in eq. (1), (continuous according our assumptions). Let L be the local martingale such that $Z = \mathcal{E}(L)$. If M is a local $\mathbb{P}$-martingale then $\tilde{M} = M - [L, M]$ is a local $\mathbb{Q}$–martingale. In particular if B is a $(\mathcal{F}_\bullet, \mathbb{P})$–Brownian motion then $\tilde{B} = B - [B, L]$ is a $(\mathcal{F}_\bullet, \mathbb{Q})$–Brownian motion.*

Proof. By Itô formula

$$\mathrm{d}(\tilde{M}Z) = Z\mathrm{d}\tilde{M} + \tilde{M}\mathrm{d}Z + \mathrm{d}[\tilde{M}, Z] = Z\mathrm{d}M - Z\mathrm{d}[L, M] + \tilde{M}\mathrm{d}Z + Z\mathrm{d}[M, L] = Z\mathrm{d}M + \tilde{M}\mathrm{d}Z \tag{8}$$

since $[\tilde{M}, Z] = [M, Z]$ due to the fact that $\tilde{M} - M$ is of bounded variation and $\mathrm{d}[M, Z] = Z\mathrm{d}[M, L]$ since $\mathrm{d}Z = Z\mathrm{d}L$ by the definition (6) of $\mathcal{E}(L)$. Then $\tilde{M}Z$ is a local $\mathbb{P}$–martingale given that the r.h.s. of (8) is the sum of two stochastic integrals wrt. the local $\mathbb{P}$–martingales M and Z. By (5) we conclude that $\tilde{M}$ is a local $\mathbb{Q}$-martingale. We remark that $[\tilde{B}, \tilde{B}]_t = [B, B]_t = t$ for all $t \geqslant 0$ so by Levy's theorem $\tilde{B}$ is a $(\mathcal{F}_\bullet, \mathbb{Q})$–Brownian motion. □

Remark 5. By Girsanov theorem two equivalent probability measures $\mathbb{P}$ and $\mathbb{Q}$ agree in classifing the same process X as a semimartingale. Indeed if X is a $\mathbb{P}$–semimartingale with decomposition $X = X_0 + M + V$ then X is also a $\mathbb{Q}$–semimartingale with decomposition $X = X_0 + \tilde{M} + \tilde{V}$ where $\tilde{V} = V + [L, M]$. On the other hand, note that, again by Girsanov, $\tilde{L} = -L + [L]$ is a local martingale under $\mathbb{Q}$ and

$$\frac{\mathrm{d}\mathbb{P}}{\mathrm{d}\mathbb{Q}}\Big|_{\mathcal{F}_t} = \tilde{Z}_t = Z_t^{-1} = \exp(-L + [L] / 2) = \exp(\tilde{L} - [\tilde{L}] / 2) = \mathcal{E}(\tilde{L})$$

is a $\mathbb{Q}$–martingale. Then if $\tilde{N}$ is a local $\mathbb{Q}$–martingale, the process $N: = \tilde{N} - [\tilde{N}, \tilde{L}] = \tilde{N} + [\tilde{N}, L] = \tilde{N} + [N, L]$ is a $\mathbb{P}$–martingale. In particular :

$$M \text{ is a local } \mathbb{P}\text{–martingale iff } M - [M, L] \text{ is a local } \mathbb{Q}\text{–martingale.}$$

Remark 6. *(Finite horizon)* We can replace an infinite time horizon with a finite one, $t_* < +\infty$ and require that $\mathbb{P}$ and $\mathbb{Q}$ are equivalent on $\mathcal{F}_{t_*}$ and $(Z_t)_{t\in[0,t_*]}$ is a continuous $\mathbb{P}$–martingale for which there exists a local $\mathbb{P}$–martingale $(L_t)_{t\in[0,t_*]}$ such that $Z = \mathcal{E}(L)$, etc...

Example 7. (Brownian motion with constant drift) Let X be a d–dimensional $\mathbb{P}$–Brownian motion and $\gamma \in \mathbb{R}^d$ a fixed vector. Consider the martingale

$$Z_t^\gamma := \exp\left(\gamma.X_t - \frac{1}{2}|\gamma|^2 t\right) = \mathcal{E}(\gamma.X)_t, \qquad t \geqslant 0$$

and for any $t \geqslant 0$ the measure $\mathbb{P}_t^\gamma$ defined on $(\Omega, \mathcal{F}_t, (\mathcal{F}_s)_{\leqslant s \leqslant t})$ by $\mathrm{d}\mathbb{P}_t^\gamma = Z_t^\gamma \mathrm{d}\mathbb{P}|_{\mathcal{F}_t}$. By Girsanov's theorem we have that, the process

$$\tilde{X}_s = X_s - [L, X]_s = X_s - [\gamma.X, X]_s = X_s - \gamma s, \qquad s \in [0, t]$$

is a d–dimensional $\mathbb{P}_t^\gamma$–Brownian motion, so under $\mathbb{P}_t^\gamma$ the process X is a Brownian motion with drift γ. Note that the family of measures $((\Omega, \mathcal{F}_t, \mathbb{P}_t^\gamma))_{t\geqslant 0}$ is consistent: $\mathbb{P}_t^\gamma|_{\mathcal{F}_s} = \mathbb{P}_s^\gamma$ for all $0 \leqslant s \leqslant t$. By Kolmogorov extension theorem there exists a unique measure $\mathbb{P}^\gamma$ on $(\Omega, \mathcal{F}_\infty)$ such that $\mathbb{P}^\gamma|_{\mathcal{F}_t} = \mathbb{P}_t^\gamma$ for all $t \geqslant 0$. Now, the $\mathcal{F}_\infty$ measurable event

$$\lim_{s\to+\infty} \frac{(X_s - \gamma s)}{s} = \gamma,$$

has $\mathbb{P}$ probability 1 (e.g. for the law of iterated logarithm applied to the $\mathbb{P}$–BM X) while it has $\mathbb{P}^\gamma$ probability 0 since $s \mapsto X_s - \gamma s$ is a $\mathbb{P}^\gamma$–Brownian motion.

1.2 Doob's h–transform

Fix a finite time horizon $I = [0, t_*]$ and consider a measure $\mathbb{P}$ on $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in I})$ with $\mathcal{F} = \mathcal{F}_{t_*}$. Let $(X_t)_{t\in I}$ an $\mathbb{R}^d$–valued Itô process with drift b, diffusion coefficient σ and generator $\mathcal{L} = b.\nabla + \frac{1}{2}\sigma\sigma^T\colon \nabla^2$. We denote by B the m–dimensional Brownian motion driving X (d, m are arbitrary integers). A very simple and natural source of equivalent change of measures for $\mathbb{P}$ is obtained from functions $h \in C^{1,2}(I \times \mathbb{R}^d; \mathbb{R}_{>0})$ such that the process $(Z_t := h(t, X_t))_{t\in I}$ is a martingale. In this case we can define $\mathbb{Q}$ by $\mathrm{d}\mathbb{Q} = Z\mathrm{d}\mathbb{P}$. By Itô formula, the function Z satisfies

$$\mathrm{d}Z_t = \sigma_t^T \nabla h(t, X_t).\mathrm{d}B_t + (\partial_t + \mathcal{L})h(t, X_t)\mathrm{d}t$$

but by the assumed martingale property the bounded variation part in the semimartingale decomposition of Z must vanish, so

$$(\partial_t + \mathcal{L})h(t, X_t) = 0$$

for almost every $t \in I$, $\mathbb{P}$-a.s. In order to have a properly normalized probability measure we need that $Z_0 = h(0, X_0) = 1$. Now, $\mathrm{d}L_t = \mathrm{d}Z_t / Z_t = (\sigma_t^T \nabla h(t, X_t)) / h(t, X_t) \mathrm{d}B_t = \sigma_t^T \nabla \log h(t, X_t) \mathrm{d}B_t$ so

$$Z_t = \exp\left(\int_0^t \sigma_s^T \nabla \log h(s, X_s) \mathrm{d}B_s - \frac{1}{2} \int_0^t |\sigma_s^T \nabla \log h(s, X_s)|^2 \mathrm{d}s \right)$$

and in particular, by Girsanov theorem, under $\mathbb{Q}$ the ($\mathbb{R}^m$–valued) process B is a semimartingale given by

$$B_t = \tilde{B}_t + [B, L]_t = \tilde{B}_t + \sigma_t^T \nabla \log h(t, X_t) \mathrm{d}t$$

where $\tilde{B}$ is an $\mathbb{R}^m$–valued $\mathbb{P}$–Brownian motion (on the interval I). As a consequence the Itô process X has decomposition

$$\mathrm{d}X_t = b_t \,\mathrm{d}t + \sigma_t \,\mathrm{d}B = (b_t + \sigma_t \sigma_t^T \nabla \log h(t, X_t)) \,\mathrm{d}t + \sigma_t \,\mathrm{d}\tilde{B} = \tilde{b}_t \,\mathrm{d}t + \sigma_t \,\mathrm{d}\tilde{B}.$$

This shows that under $\mathbb{P}$ the process X remains an Itô process with the same diffusion coefficient but a different drift $\tilde{b}$ given by

$$\tilde{b}_t = b_t + \sigma_t \sigma_t^T \nabla \log h(t, X_t), \qquad t \in I.$$

This construction is called *Doob's h-transform* and was originally introduced in the context of Markov processes. In the case of Itô diffusions (i.e. where the drift and diffusion coefficients b_t, σ_t are deterministic functions of X_t) the h–transform gives a transformation of the associated martingale problems. Indeed from the previous discussion we see that if $(X, \mathbb{P})$ is a solution of the martingale problem $\mathrm{MP}(x_0, b, \sigma\sigma^T)$ then $(X, \mathbb{Q})$ is a solution of the martingale problem $\mathrm{MP}(x_0, \tilde{b}, \sigma\sigma^T)$.

Exercise 1. Prove that, more generally, the h-transform gives a transformation of martingale problems from $\mathrm{MP}(x_0, b, a)$ to $\mathrm{MP}(x_0, \tilde{b}, a)$. (That is, reproduce the above argument without relying on the Ito decomposition of X)

The Brownian motion with drift introduced in Example 7 is a particular case of this transform where the function h is given by

$$h(t, x) = \exp(\gamma . x - |\gamma| t / 2), \qquad t \geqslant 0, x \in \mathbb{R}^d.$$

Very interesting cases of h–transforms arise if we allow the interval $I = [0, t_*)$ (or $[0, +\infty)$) to be open at the right endpoint, i.e. if we do not require the measure $\mathbb{Q}$ to be equivalent to $\mathbb{P}$ on $\mathcal{F}_{t_*}$ but given by $\mathrm{d}\mathbb{Q}|_{\mathcal{F}_t} = Z_t \,\mathrm{d}\mathbb{Q}|_{\mathcal{F}_t}$ for all $t \in I$. In this case we ask that $(Z_t)_{t \in I}$ be striclty positive continuous martingale in I but we do not pose any restriction on Z_{t_*}.

1.3 Diffusion bridges

Using Doob's h–transform we can describe very effectively the behavior of a Markovian Itô diffusion X with values in $\mathbb{R}^d$ conditioned to reach some state $x_1 \in \mathbb{R}^d$ at a given time (which we take to be 1). The interesting fact is that this kind of conditioning is singular since usually the event is of probability 0 wrt. the law of the unconditioned diffusion. Let $I = [0, 1)$ and assume that the Itô diffusion X is a Markov process with transition density

$$\mathbb{P}(X_t \in \mathrm{d}y | X_s = x) = p(s, x; t, y) \mathrm{d}y.$$

In this case we can define the function

$$g^{x_1}(s,x) := \frac{p(s,x;1,x_1)}{p(0,x_0;1,x_1)}, \qquad s\in I, \quad x\in\mathbb{R}^d$$

and assume that g is $C^{1,2}(I\times\mathbb{R}^d;\mathbb{R}_{>0})$. Let $Z_t^{x_1} = g^{x_1}(t, X_t)$ for $t\in I$ and observe that, by construction, $Z_t^{x_1}$ is a martingale since

$$\mathbb{E}[Z_t^{x_1}|\mathcal{F}_s] = \mathbb{E}[Z_t^{x_1}|X_s] = \int \frac{p(t,y;1,x_1)p(s,X_s;t,y)}{p(0,x_0;1,x_1)}\mathrm{d}y = \frac{p(s,X_s;1,x_1)}{p(0,x_0;1,x_1)} = Z_s^{x_1}, \qquad 0\leqslant s\leqslant t<1,$$

by the Markov property and the Chapman–Kolmogorov equation. Define the measure $\mathbb{P}^{x_1}$ on $\mathcal{F}_1$ as the extension of the family $(\mathbb{P}_t^{x_1})_{t\in I}$ where $\mathrm{d}\mathbb{P}_t^{x_1} = Z_t\,\mathrm{d}\mathbb{P}|_{\mathcal{F}_t}$ and observe that for all $0<t_1<\cdots<t_n<1$ and bounded test function f we have

$$\int f(X_{t_1},\ldots,X_{t_n},x_1)p(t_n,X_{t_n};1,x_1)\mathrm{d}x_1 = \mathbb{E}_{\mathbb{P}}[f(X_{t_1},\ldots,X_{t_n},X_1)|\mathcal{F}_{t_n}]$$

by Markov property. Then taking expectation wrt $\mathbb{P}$ we obtain

$$\mathbb{E}_{\mathbb{P}}[f(X_{t_1},\ldots,X_{t_n},X_1)] = \int \mathbb{E}_{\mathbb{P}^{x_1}}[f(X_{t_1},\ldots,X_{t_n},x_1)]p(0,x_0;1,x_1)\mathrm{d}x_1$$

which shows that letting $\Phi_f(y) := \mathbb{E}_{\mathbb{P}^y}[f(X_{t_1},\ldots,X_{t_n},x_1)]$ for $y\in\mathbb{R}^d$, we have

$$\mathbb{E}_{\mathbb{P}}[f(X_{t_1},\ldots,X_{t_n},X_1)r(X_1)] = \mathbb{E}_{\mathbb{P}}[\Phi_f(X_1)r(X_1)]$$

for any bounded measurable function r. In particular $\Phi_f(X_1) = \mathbb{E}_{\mathbb{P}}[f(X_{t_1},\ldots,X_{t_n},X_1)|X_1]$ and Φ_f is a regular conditional probability for $\mathbb{P}$ given $\sigma(X_1)$. We conclude that $\mathbb{P}^{x_1}$ is obtained by h–transforming $\mathbb{P}$ via the function g, in particular that under the measure $\mathbb{P}^{x_1}$ the Ito diffusion X has a new drift b^{x_1} given by :

$$b^{x_1}(t,X_t) = b(t,X_t) + (\sigma\sigma^T)(t,X_t)\nabla\log p(t,X_t;1,x_1), \qquad t\in I.$$

Usually the additional logarithmic derivative of the transition density become singular as $t\to 1$. This accounts for the fact that the diffusion must satisfy $X_t\to x_1$ when $t\to 1$ and the martingale part of the diffusion has to be counterbalances by a the strong drift in order for this to happen with probability one.

Exercise 2. Compute the drift in the case of X being a Brownian motion in $\mathbb{R}^d$.

1.4 Diffusions conditioned to stay in a given domain

We want to give an idea of the method used by Pinsky [3] to study a Markovian diffusion process X on $\mathbb{R}^d$ conditioned to stay in a given subdomain $D\subseteq\mathbb{R}^d$. For more details please refer to the original paper.

Let $D \subseteq \mathbb{R}^d$ an open bounded and connected set and let $\tau_D = \inf\{t \geqslant 0 \colon X_t \notin D\}$. Let $(X, \mathbb{P}_x)$ be a Markov process satisfying the SDE

$$\mathrm{d}X_t = b(X_t)\mathrm{d}t + \sigma(X_t)\mathrm{d}B_t, \qquad X_0 = x.$$

where B in a Brownian motion in $\mathbb{R}^m$. Let $\mathcal{L} = b.\nabla + \frac{1}{2}\sigma\sigma^T \colon \nabla^2$ the generator of X. Fix $T > 0$ and define the measure $\mathbb{Q}_x^T(\cdot) = \mathbb{P}_x(\cdot|\tau_D > T)$ for the process X conditioned to stay in D up to time T. We assume that the function

$$g^T(x) = \mathbb{P}_x(\tau_D > T) \qquad x \in D$$

is in $C^2(D)$ and that

$$\lim_{T \to \infty} \frac{\nabla g^T(x)}{g^T(x)} = \frac{\nabla \varphi_0(x)}{\varphi_0(x)}$$

uniformly on compact subsets of D. Here φ_0 is the eigenfunction of $\mathcal{L}$ with Dirichlet boundary conditions corresponding to the real simple eigenvalue λ_0. We will assume that $\varphi_0 > 0$ on D and that $\varphi_0 = 0$ on ∂D. Let M^T be the process $M_t^T := g^{T-t}(X_{t\wedge\tau_D}) \,/\, g^T(X_0)$, by the Markov property of X we have

$$\mathbb{E}_{\mathbb{Q}_x^T}[H] = \frac{\mathbb{E}_x[H\, \mathbb{I}_{\tau_D > T}]}{g^T(x)} = \frac{\mathbb{E}_x[H\, g^T(X_t)\mathbb{I}_{\tau_D > t}]}{g^T(x)} = \frac{\mathbb{E}_x[H\, g^T(X_{t\wedge\tau_D})]}{g^T(x)} = \mathbb{E}_x[H\, M_t^T]$$

for any $t < T$ and any r.v. H bounded and $\mathcal{F}_t$ measurable. Moreover

$$\mathbb{E}[M_t^{T-t}|\mathcal{F}_s] = \mathbb{E}[\mathbb{P}_{X_t}(\tau_D > T - t)\mathbb{I}_{\tau_D > t}|\mathcal{F}_s] = \mathbb{P}_{X_{s\wedge\tau_D}}(\tau_D > T - s) = g^{T-s}(X_{s\wedge\tau_D}) = M_s^T, \quad t \geqslant s \geqslant 0.$$

Note that under $\mathbb{Q}_x^T$ we have $\mathbb{Q}_x^T(\tau_D \leqslant t) = 0$ so that under $\mathbb{Q}_x^T$ the process X takes values in D. By Doob's h-transform this implies that, under $\mathbb{Q}_x^T$ the process X is an Itô diffusion in D with diffusion matrix $\sigma|_D$ and drift

$$\tilde{b}^T(x) = b(x) + \frac{\nabla g^T(x)}{g^T(x)}, \qquad x \in D.$$

Under some technical assumptions on the behaviour of g^T as $T \to \infty$ it is possible to show that on any bounded interval $[0, S]$ the process $(X, \mathbb{Q}_x^T)$ converges in law, as $T \to \infty$, to a weak solution $(X, \mathbb{Q}_x)$ of the SDE with diffusion matrix σ and drift

$$\tilde{b}(x) = \lim_{T \to \infty} \tilde{b}^T(x) = b(x) + (\sigma\sigma^T)(x)\frac{\nabla \varphi_0(x)}{\varphi_0(x)}, \qquad x \in D.$$

The corresponding law $(X, \mathbb{Q}_x)$ can be understood as the law of original process conditioned never to leave D.

Example 8. (Brownian motion conditioned to stay positive) Let X be a Brownian motion on $\mathbb{R}$ starting at $x_0 \in (0, L)$ for some $L > 0$. Let $\mathbb{Q}$ be the measure obtained by conditioning the Brownian motion to hit L before 0:

$$\mathrm{d}\mathbb{Q}|_{\mathcal{F}_\infty} = \frac{\mathbb{I}_A \mathrm{d}\mathbb{P}|_{\mathcal{F}_\infty}}{\mathbb{P}(A)},$$

where $A = \{T = +\infty \text{ or } X_T = L\}$ and $T = \inf\{t \geqslant 0: X_t \notin (0, L)\}$. Note that $\mathbb{P}(T = +\infty) = 0$. Letting $h(x) = (x / x_0)$ the process $Z_t = h(X_t^T)$ is a martingale starting at 1 and $\frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}}|_{\mathcal{F}_t} = Z_t$ for all $t \geqslant 0$. By Girsanov theorem under $\mathbb{Q}$ the process X satisfies the SDE (on a random interval, until $X_t = L$ for the first time)

$$\mathrm{d}X_t = \mathrm{d}\tilde{X}_t + \frac{\mathrm{d}t}{X_t}, \qquad t \in [0, T]$$

starting at x_0, where $\tilde{X}$ is a $\mathbb{Q}$-Brownian motion on $[0, T]$. In particular this solution to the SDE never touches 0. Moreover as $L \to \infty$ we have $T \to \infty$ $\mathbb{P}$–p.s.. This suggests to define $\mathbb{Q}$ by $\frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}}|_{\mathcal{F}_t} = X_t^{T_0} / x_0 \geqslant 0$ for all $t \geqslant 0$ where $T_0 = \inf\{t \geqslant 0: X_t < 0\}$. Now, X^{T_0} is a positive martingale (why?) so $\mathbb{Q}$ is well defined, but singular wrt. $\mathbb{P}$ on $\mathcal{F}_\infty$ since (by recurrence of the BM) $\mathbb{P}(T_0 < \infty) = 1$ while $\mathbb{Q}(T_0 < \infty) = 0$ by definition. In this case the process X satisfies the SDE

$$\mathrm{d}X_t = \mathrm{d}\tilde{X}_t + \frac{\mathrm{d}t}{X_t}, \qquad t \geqslant 0,$$

where now $\tilde{X}$ is a $\mathbb{Q}$-Brownian motion on $\mathbb{R}_{\geqslant 0}$.

Example 9. (Brownian motion conditioned to stay in $D = (0, \pi)$) Let X be a Brownian motion under $\mathbb{P}$. In this case the first Dirichlet eigenfunction of the Laplacian in D is $\varphi_0(x) = \sin(x)$ and $\lambda = 1$ the corresponding eigenvalue. Let $h(t, x) = e^t \varphi_0(x)$. Then $Z_t = h(t, X_t) / h(0, x_0)$ is a $\mathbb{P}$–martingale (by Itô formula) and we can define the measure $\mathbb{Q}$ with density Z_t wrt $\mathbb{P}$ on $\mathcal{F}_t$. Under $\mathbb{Q}$ the process X satisfies the SDE

$$\mathrm{d}X_t = \mathrm{d}B_t + \frac{\cos(X_t)}{\sin(X_t)} \mathrm{d}t.$$

Example 10. (Brownian motion in a Weyl chamber). Let $X = (X^1, \ldots, X^n)$ be a n–dimensional Brownian motion starting at $x_0 \in S$ where $S \subseteq \mathbb{R}^n$ is the Weyl chamber, that is the set

$$S = \{x = (x^1, \ldots, x^n) \in \mathbb{R}^n: x^1 < x^2 < \cdots < x^n\}.$$

Consider the function $h(x) = \prod_{i<j} (x^j - x^i) / \prod_{i<j} (x_0^j - x_0^i)$ and let $T_0 = \inf\{t \geqslant 0: h(X_t) = 0\}$. One can check that h is harmonic in S. Performing the h transform with this function we obtain a system of interacting Brownian motions on $\mathbb{R}$ satisfying, under $\mathbb{Q}$:

$$\mathrm{d}X_t^i = \sum_{j \neq i} \frac{1}{X^i - X^j} \mathrm{d}t + \mathrm{d}B_t^i, \qquad i = 1, \ldots, n, \quad t \geqslant 0.$$

In particular this shows that under $\mathbb{Q}$ the processes $(X^i)_i$ never intersects and preserve their linear order. This process is called Dyson's Brownian motion [1] and coincide with the process describing the evolution of eigenvalues of a natural continuous diffusion in the space of symmetric $n \times n$ matrices.

1.5 Exponential tilting

As seen in the previous examples and in many applications we are required to study the measures $\mathbb{P}^L$ on $(\Omega, \mathcal{F}_\infty)$ obtained from $\mathbb{P}$ via exponential "tilting" with a *given* local continuous $\mathbb{P}$–martingale L (started at 0):

$$\mathrm{d}\mathbb{P}^L|_{\mathcal{F}_t} = \mathcal{E}(L)_t \,\mathrm{d}\mathbb{P}|_{\mathcal{F}_t}, \qquad t \geqslant 0. \tag{9}$$

In order for this construction to make sense and $\mathbb{P}^L$ be well defined, we need $(\mathcal{E}(L)_t)_{t\geqslant 0}$ to be a (true) martingale and not just a local one (otherwise we cannot even guarantee that $\mathbb{P}^L(\Omega) = 1$). Moreover the fact that the family of measures $\mathrm{d}\mathbb{P}^L_t = \mathcal{E}(L)_t \,\mathrm{d}\mathbb{P}|_{\mathcal{F}_t}$ on $(\Omega, \mathcal{F}_t)$ indentifies a unique measure $\mathbb{P}^L$ on $(\Omega, \mathcal{F}_\infty)$ via $\mathbb{P}^L|_{\mathcal{F}_t} = \mathbb{P}^L_t$ is due to Kolmogorov–Daniell extension theorem: indeed the family is consistent $\mathbb{P}^L_t|_{\mathcal{F}_s} = \mathbb{P}^L_s$ iff $\mathcal{E}(L)$ is a martingale.

In a finite horizon $I = [0, t_*]$ we need to ensure that $\mathbb{E}[\mathcal{E}(L)_{t_*}] = 1$.

Lemma 11. *A positive local martingale M such that $\mathbb{E}[M_0] < \infty$ is a supermartingale and it converges a.s. as $t \to \infty$ to a limit $M_\infty \geqslant 0$ such that $\mathbb{E}[M_\infty] \leqslant \mathbb{E}[M_0]$. If $\mathbb{E}[M_\infty] = \mathbb{E}[M_0]$ then M is a UI martingale.*

Proof. Let $(T_n)_{n\geqslant 0}$ a reducing family for $(M_t - M_0)_{t\geqslant 0}$, then for all $0 \leqslant s \leqslant t$

$$\mathbb{E}[M_t^{T_n} - M_0|\mathcal{F}_s] = \mathbb{E}[M_t^{T_n}|\mathcal{F}_s] - M_0 = M_s^{T_n} - M_0.$$

By the conditional form of Fatou's lemma (and the integrability of M_0)

$$\mathbb{E}[M_t|\mathcal{F}_s] = \mathbb{E}\Big[\liminf_n M_t^{T_n}\Big|\mathcal{F}_s\Big] \leqslant \liminf_n \mathbb{E}[M_t^{T_n}|\mathcal{F}_s] = M_s.$$

We will assume (without proof) that a positive continuous supermartingale converges a.s. The next exercise completes the proof. □

Exercise 3. Let M be a positive continuous supermartingale such that $\mathbb{E}[M_0] < \infty$. Let $M_\infty = \lim_{t\to\infty} M_t$ (assumed to exist $\mathbb{P}$–a.s.). Show that if $\mathbb{E}[M_\infty] = \mathbb{E}[M_0]$ then M is a UI martingale. *[Hint: prove that $\mathbb{E}[M_\infty|\mathcal{F}_t] \leqslant M_t$ and that $\mathbb{E}[M_t] = \mathbb{E}[M_0]$ and conclude.]*

The next results describes *sufficient* conditions under which we can ensure the crucial property $\mathbb{E}[\mathcal{E}(L)_\infty] = 1$ when $L_0 = 0$. As just shown in Lemma 11 this implies that $(\mathcal{E}(L)_t)_{t\geqslant 0}$ is a UI martingale. Condition (11) below is called Novikov's condition. Condition (10) is due to Krylov [2] from which the proof below is taken.

Theorem 12. (Novikov, Krylov) *Let L be a local martingale starting at 0 and assume that*

$$\lim_{\varepsilon \downarrow 0} \varepsilon \log \mathbb{E}[\exp((1-\varepsilon)[L]_\infty/2)] = 0 \tag{10}$$

then $\mathbb{E}[\mathcal{E}(L)_\infty] = 1$. In particular this holds if

$$\mathbb{E}[\exp([L]_\infty/2)] < \infty. \tag{11}$$

Proof. We will start by proving the following weaker statement:

$$\exists p_0 > 1 \colon \mathbb{E}[\exp(p_0[L]_\infty/2)] < +\infty \Longrightarrow \mathbb{E}[\mathcal{E}(L)_\infty] = 1.$$

Let $(T_n)_{n\geqslant 1}$ be a family of stopping times reducing L and such that $T_n \leqslant n$ for all $n \geqslant 1$. Then for any $r \geqslant 1$ we have

$$\mathbb{E}[(\mathcal{E}(L^{T_n})_\infty)^r] = \mathbb{E}[\exp(r L_\infty^{T_n} - r\,[L^{T_n}]_\infty/2)] = \mathbb{E}[\exp(r L_\infty^{T_n} - r^2\,p[L^{T_n}]_\infty/2)\exp((r^2 p - r)[L^{T_n}]_\infty/2)]$$

by Hölder inequality (with $p, q > 1$ and $p^{-1} + q^{-1} = 1$)

$$\mathbb{E}[(\mathcal{E}(L^{T_n})_\infty)^r] \leqslant (\mathbb{E}[\mathcal{E}(r p L^{T_n})_\infty])^{1/p} (\mathbb{E}[\exp(\ \gamma(r,p)[L^{T_n}]_\infty/2)])^{1/q},$$

where

$$\gamma(r,p) := q(r^2 p - r) = \frac{r^2 p - r}{1 - p^{-1}} = p\,r\frac{r p - 1}{p - 1} > 1.$$

When $r \to 1+$, $\gamma(r,p) \to p+$ so we can choose $1 < p < p_0$ and $r = 1 + \varepsilon$ for sufficiently small $\varepsilon > 0$ to ensure that $\gamma(r,p) \leqslant p_0$. With this choice, using the fact that $\mathcal{E}(r p L^{T_n})$ is a martingale, that $[L^{T_n}]_\infty \leqslant [L]_n \leqslant [L]_\infty$ and the hypothesis of (11), we get

$$\mathbb{E}[(\mathcal{E}(L^{T_n})_\infty)^r] \leqslant (\mathbb{E}[\exp(\ \gamma(r,p)[L]_\infty/2)])^{1/q} \leqslant (\mathbb{E}[\exp(p_0[L]_\infty/2)])^{1/q} < +\infty.$$

Since $r > 1$, the fact that $\sup_n \mathbb{E}[(\mathcal{E}(L^{T_n})_\infty)^r] < +\infty$ implies that the family $(\mathcal{E}(L^{T_n})_\infty)_n$ is uniformly integrable for any $t \geqslant 0$. From this we conclude that $\mathbb{E}(\mathcal{E}(L)_\infty) = \mathbb{E}[\lim_n \mathcal{E}(L^{T_n})_\infty] = \lim_n \mathbb{E}[\,\mathcal{E}(L^{T_n})_\infty] = 1$ so (11) is proven.

Let us go back to the proof of the Lemma. Take $\varepsilon \in (0,1)$ and observe that letting $p_0 = (1+\varepsilon)^2 > 1$ we have

$$\mathbb{E}[\exp(p_0[(1-\varepsilon)L]_\infty/2)] = \mathbb{E}[\exp((1-\varepsilon^2)[L]_\infty/2)] < \infty.$$

which by (11) implies that $\mathbb{E}[\mathcal{E}((1-\varepsilon)L)_\infty] = 1$. By Hölder inequality,

$$1 = \mathbb{E}[\mathcal{E}((1-\varepsilon)L)_\infty] = \mathbb{E}[\exp((1-\varepsilon)(L_\infty - [L]_\infty/2))\exp((1-\varepsilon)\varepsilon[L]_\infty/2)]$$

$$\leqslant (\mathbb{E}[\mathcal{E}(L)_\infty])^{(1-\varepsilon)}\,(\mathbb{E}[\exp((1-\varepsilon)[L]_\infty/2)])^{\varepsilon}.$$

Letting $\varepsilon \to 0$ we obtain $1 \leqslant \mathbb{E}[\mathcal{E}(L)_\infty]$ due to (10). This concludes our argument since we already know that $\mathbb{E}[\mathcal{E}(L)_\infty] \leqslant 1$ (Fatou). □

Remark 13. The conditions (10) and (11) are not necessary for $\mathcal{E}(L)$ to be a martingale. Another well known sufficient condition is Kazamaki condition

$$\mathbb{E}[\exp(L_\infty/2)] < +\infty, \tag{12}$$

which is weaker than Novikov's. For more details refer to Revuz and Yor [4].

1.6 Tilting via a Brownian local martingale

Let $(\Omega, \mathcal{F} = \mathcal{F}_\infty, \mathcal{F}_\bullet, \mathbb{P})$ carry a d–dimensional Brownian motion B and let b an adapted $\mathbb{R}^d$-valued process for which

$$\int_0^t |b_s|^2 \mathrm{d}s < +\infty, \qquad \mathbb{P} - a.s. \qquad \forall t \geqslant 0.$$

Let L^b be the scalar continous local martingale

$$L_t^b := \int_0^t b_s . \mathrm{d}B_s, \qquad t \geqslant 0,$$

assume that $\mathcal{E}(L^b)$ is a martingale and define $\mathbb{P}^b := \mathbb{P}^L$ as in eq. (9). The process B satisfies the equation

$$\mathrm{d}B_t = b_t \,\mathrm{d}t + \mathrm{d}W_t, \qquad t \geqslant 0,$$

where W is a $\mathbb{P}^b$–Brownian motion.

Example 14. (Solutions to SDEs) Exponential tilting via the martingale L^b is useful to construct (weak) solutions to SDEs with a general class of drift coefficients. It will be convenient to assume that $\Omega = C(R_{\geqslant 0}; \mathbb{R}^d)$, that $\mathbb{P}$ is the d–dimensional Wiener measure and that X is the canonical process on Ω and that the filtration $\mathcal{F}_\bullet$ is generated by X. We consider a predictable $\mathbb{R}^d$-valued process $\hat{b}$ given by a function $b: \mathbb{R}_{\geqslant 0} \times \Omega \to \mathbb{R}^d$ via $\hat{b}_t(\omega) = b_t(X(\omega))$ and we denote it $\hat{b} =: b(X)$. By tilting $\mathbb{P}$ via $L^{b(X)}$ we obtain that, under $\mathbb{P}^b$ the process X is a solution of the SDE

$$\mathrm{d}X_t = b_t(X) + \mathrm{d}W_t, \qquad t \geqslant 0,$$

provided $\mathcal{E}(L^{b(X)})$ is a martingale. A natural condition on b which guaratees the martingale property is

$$|b_t(x)| \leqslant C_t(1 + \|x\|_{\infty,[0,t]}), \qquad t \geqslant 0, x \in \Omega \tag{13}$$

where $C_t < +\infty$ for all $t \geqslant 0$ (see next exercise).

Exercise 4.

a) Prove that if

$$|b_t(x)| \leqslant C_t(1 + |x_t|), \qquad t \geqslant 0, x \in \Omega,$$

then Novikov's condition holds for all $t \geqslant 0$, i.e.

$$\mathbb{E}\left[\exp\left(\int_0^t |b_s(X)|^2 \mathrm{d}s\right)\right] < +\infty, \qquad t \geqslant 0.$$

[Hint: show that the condition holds for t small enough and then use the Markov property to extend to all t]

b) Prove that

$$\mathbb{P}(\|X\|_{[0,t]} > r) \leqslant C e^{-r^2/2t} \qquad t \geqslant 0, r \geqslant 0.$$

[Hint: use Doob's inequality for the submartingale $e^{\lambda X_t^i}$ and optimize over $\lambda > 0$]

c) Prove the same result as in (a) under the more general assumption (13). [Hint: for small time use (b) to estimate the size of the maximum of the Brownian motion, extend to all times via the appropriate Markov process]

More generally, solutions to SDEs with coefficients b, σ (in general path dependent) are transformed to solutions:

Corollary 15. ***(Drift transformation of SDEs)*** *If* $(X, B, \mathbb{P})$ *is a weak solution to*

$$\mathrm{d}X_t = b_t(X)\mathrm{d}t + \sigma_t(X)\mathrm{d}B_t$$

and $C_t = c_t(X)$ *is such that* $\mathcal{E}(L^C)$ *is a martingale then* $(X, \tilde{B}, \mathbb{P}^C)$ *is a weak solution to*

$$\mathrm{d}X_t = \tilde{b}_t(X)\mathrm{d}t + \sigma_t(X)\mathrm{d}\tilde{B}_t$$

where the new drift is given by $\tilde{b} = b + \sigma c$ *and* $\mathrm{d}\tilde{B}_t = \mathrm{d}B_t - c_t(X)\mathrm{d}t$.

Exercise 5. Generalise Girsanov transformation to a martingale problem $\mathrm{MP}(x_0, b, a)$.

Exercise 6. Use Girsanov transform to prove the uniqueness in law of the weak solution of the SDE

$$\mathrm{d}X_t = b_t(X)\mathrm{d}t + \mathrm{d}B_t$$

where $b: \mathbb{R}_{\geqslant 0} \times C(\mathbb{R}_{\geqslant 0}; \mathbb{R}^d) \to \mathbb{R}^d$ is a bounded, previsible drift.

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