Stochastic Analysis – Course note 3

Version 1.3. Last revision 16/6/2016. Please send remarks and corrections to gubinelli@iam.uni-bonn.de.

1 The Brownian martingale representation theorem

The aim of this note is to prove a nice and useful result about representation of martingales in a Brownian filtration. In all the section we will assume that $(\Omega, \mathcal{F}, \mathcal{F}_\bullet, \mathbb{P})$ is a filtered probability space endowed with a d-dimensional Brownian motion X and such that $\mathcal{F}_\bullet$ is the canonical filtration of X ($\mathbb{P}$-completed, right–continuous) and that $\mathcal{F} = \mathcal{F}_\infty$.

We want to prove that

Theorem 1. *Let $\Phi \in L^2(\Omega, \mathcal{F}, \mathbb{P})$ then there exists a unique predictable process $F \in L^2_{\mathcal{P}}(\mathbb{R}_+ \times \Omega; \mathbb{R}^d)$ such that*

$$\mathbb{E}[\Phi|\mathcal{F}_t] = \mathbb{E}[\Phi] + \int_0^t F_s . \mathrm{d}X_s, \qquad t \geqslant 0.$$

We will present a "Markovian" proof of this result taken from [6] and inspired by the work of Meyer [5]. We need some preliminary result. Let P the transition operator of the Brownian motion:

$$P_t f(x) := \int_{\mathbb{R}^d} f(y) \frac{e^{-\frac{|x-y|^2}{2t}}}{(2\pi t)^{d/2}} \mathrm{d}y, \qquad x \in \mathbb{R}^d, t > 0$$

and U^α the resolvent operator

$$U^\alpha f(x) := \int_0^\infty e^{-\alpha s} P_s f(x) \, \mathrm{d}\, s, \qquad x \in \mathbb{R}^d, \alpha > 0.$$

Note that if $f \in C_b(\mathbb{R}^d)$ then $(\alpha - \Delta) U^\alpha f = f$. Now recall the Stone–Weierstrass theorem.

Theorem 2. (Stone–Weierstrass) *Suppose $\mathcal{X}$ is a compact Hausdorff space and $\mathcal{A}$ is a subalgebra of $C(\mathcal{X})$ which contains a non-zero constant function. Then $\mathcal{A}$ is dense in $C(\mathcal{X})$ if and only if it separates points.*

The following key lemma, giving the density in $L^p(\Omega, \mathcal{F}, \mathbb{P})$ of a special algebra of random variables, is due to Meyer.

Lemma 3. *Let $\mathcal{C} \subset L^1(\Omega, \mathcal{F}, \mathbb{P})$ be the algebra generated by functions of the form*

$$\Phi^\alpha(f) := \int_0^\infty e^{-\alpha s} f(X_s) \, \mathrm{d}\, s$$

for $\alpha > 0$ and $f \in C_0^\infty(\mathbb{R}^d)$ (compactly supported smooth functions). Then $\mathcal{C}$ is dense in $L^p(\Omega, \mathcal{F}, \mathbb{P})$ for $p \geqslant 1$.

Proof. *Step 1. Restrict to $\mathcal{F}_T$.* Let $F \in L^p(\Omega, \mathcal{F}, \mathbb{P})$ and $F_T = \mathbb{E}[F|\mathcal{F}_T]$. By Doob's convergence theorem $F_T \to F$ a.s. and in L^p when $T \to \infty$. So we can restrict ourselves to show that $\mathcal{C}$ is dense in $L^p(\Omega, \mathcal{F}_T, \mathbb{P})$ for each fixed $T > 0$.

Step 2. Restrict to tensor cylinder functions. Now we want to prove that the algebra of functions of the form

$$\prod_{i=1}^{n} g_i(X_{t_i}) \tag{1}$$

for $g_i \in C_0(\mathbb{R}^d)$ and $t_i \in [0, T]$, is dense in $L^p(\Omega, \mathcal{F}_T, \mathbb{P})$. Indeed let $\{t_i\}_{i \geqslant 0}$ a dense set of times in $[0, T]$ and $\mathcal{G}_\bullet$ the filtration given by $\mathcal{G}_n = \sigma(X_{t_i}: 0 \leqslant i \leqslant n) \vee \mathcal{F}_0$. Note that $\mathcal{G}_\infty = \mathcal{F}_T$ since by continuity of the paths of the Brownian motion each event of the form $\{X_t \in B\}$ for some fixed t and Borel B belongs to $\mathcal{G}_\infty$. For $F \in L^p(\Omega, \mathcal{F}_T, \mathbb{P})$ let $F_n = \mathbb{E}[F|\mathcal{G}_n]$. Again by Doob's theorem we have $F_n \to F$ a.s. and in L^p, morever there exists a Borel function f_n such that $F_n = f_n(X_{t_1}, \ldots, X_{t_n})$. But any such r.v. can be approximated in $L^p(\Omega, \mathcal{F}_T, \mathbb{P})$ by $\varphi_n(X_{t_1}, \ldots, X_{t_n})$ where φ_n is C_0. By the Stone–Weierstrass theorem this function can be approximated uniformly (on its compact support) by polynomials for which it is easy to see that the factorization (1) holds. In order to have a uniform approximation everywhere it is enough to multiply the polynomials by an appropriate C_0^∞ localization function which can be chosen in factorized form.

Step 3. To exponential integrals in time. By continuity of the Brownian paths we can find functions $h_i \in C_0([0, T]; \mathbb{R})$ such that

$$\prod_{i=1}^{n} \int_0^T g_i(X_s) h_i(s) \mathrm{d}s$$

approximate (1) arbitrarily well in L^p. Again by Stone–Weierstrass we can approximate each of the functions $t \mapsto h_i$ by linear combinations $\tilde{h}_i$ of exponential functions $(s \mapsto e^{-\alpha s})_{\alpha > 0}$ since these functions separate the points of $[0, T]$. This can be done so that $|h_i(t) - \tilde{h}_i(t)| \leqslant \varepsilon$ where ε can be chosen arbitrarily small. Now

$$\left| \int_0^T g_i(X_s) \big(h_i(s) - \tilde{h}_i(s) \big) \mathrm{d}s \right| \leqslant \|g\|_\infty T \varepsilon \text{CHECK!!!}$$

This estimate allows to control the error in the integral. So we conclude that we can approximate any element of $L^p(\Omega, \mathcal{F}^X, \mathbb{P})$ by linear combinations of elements of $\mathcal{C}$. □

Proof. (of Theorem 1) *Step 1. Representation of elements of $\mathcal{C}$.* Conditional expectations of elements of $\mathcal{C}$ can be computed explicitly. By the Markov property

$\mathbb{E}[\Phi^\alpha(f)|\mathcal{F}_t] = \int_0^t e^{-\alpha s} f(X_s) \mathrm{d}s + e^{-\alpha t} \int_0^\infty e^{-\alpha s} P_s f(X_t) \mathrm{d}s = \int_0^t e^{-\alpha s} f(X_s) \mathrm{d}s + e^{-\alpha t} U^\alpha f(X_t)$.

For a more general element of $\mathcal{C}$ we have

$$\prod_i \Phi^{\alpha_i}(f_i) = \sum_{\sigma\in S_n} \int_{0<s_1<\cdots<s_n} \prod_i \left[(\mathbb{I}_{s_i\leqslant t} + \mathbb{I}_{s_i>t})\, e^{-\alpha_{\sigma(i)}s_i} f_{\sigma(i)}(X_{s_i})\right] \mathrm{d}\, s_1 \cdots \mathrm{d}\, s_n$$

$$= \sum_{k=0}^{n} \sum_{\sigma\in S_n} \int_{0<s_1<\cdots<s_n} \mathbb{I}_{s_k\leqslant t, s_{k+1}>t} \prod_i \left[e^{-\alpha_{\sigma(i)}s_i} f_{\sigma(i)}(X_{s_i})\right] \mathrm{d}\, s_1 \cdots \mathrm{d}\, s_n$$

$$= \sum_{k=0}^{n} \sum_{\sigma\in S_n} V_t^{\sigma,k}(X) \int_{t<s_{k+1}<\cdots<s_n} \prod_{i=k+1}^{n} \left[e^{-\alpha_{\sigma(i)}s_i} f_{\sigma(i)}(X_{s_i})\right] \mathrm{d}\, s_{k+1} \cdots \mathrm{d}\, s_n$$

where

$$V_t^{\sigma,k}(X) := \int_{0<s_1<\cdots<s_k<t} \prod_{i=1}^{k} \left[e^{-\alpha_{\sigma(i)}s_i} f_{\sigma(i)}(X_{s_i})\right] \mathrm{d}\, s_1 \cdots \mathrm{d}\, s_k.$$

An easy computation shows that, again by the Markov property,

$$\mathbb{E}\left[\int_{t<s_{k+1}<\cdots<s_n} \prod_{i=k+1}^{n} \left[e^{-\alpha_{\sigma(i)}s_i} f_{\sigma(i)}(X_{s_i})\right] \mathrm{d}\, s_{k+1} \cdots \mathrm{d}\, s_n \middle| \mathcal{F}_t\right] =$$

$$= e^{-\alpha(\sigma,k)t}\, U^{\alpha(\sigma,k)}\left(f_{\sigma(k+1)}\, U^{\alpha(s,k+1)}(f_{\sigma(k+2)} \cdots U^{\alpha_{\sigma(n)}}(f_{\sigma(n)}))\right)(X_t)$$

where $\alpha(\sigma,k) := \alpha_{\sigma(k+1)} + \cdots + \alpha_{\sigma(n)}$. This gives the claimed explicit expression of the conditional probability:

$$M_t = \mathbb{E}\left[\prod_i \Phi^{\alpha_i}(f_i) \middle| \mathcal{F}_t\right] = \sum_{k=0}^{n} \sum_{\sigma\in S_n} V_t^{\sigma,k}(X) e^{-\alpha(\sigma,k)t}\, U^{\alpha(\sigma,k)}(H^{\sigma,k})(X_t)$$

where

$$H^{\sigma,k}(x) := f_{\sigma(k+1)}(x)\, U^{\alpha(\sigma,k+1)}\left(f_{\sigma(k+2)} \cdots U^{\alpha_{\sigma(n)}}(f_{\sigma(n)})\right)(x).$$

This formula in particular shows that M is a continuous martingale since $t \mapsto V_t^{\sigma,k}(X)$ is continuous and for any $f \in C_0^\infty$ the function $x \mapsto U^\alpha f(x)$ is C_0^∞. By Itô formula we have

$$\mathrm{d}\, U^{\alpha(\sigma,k)}(H^{\sigma,k})(X_t) = \nabla U^{\alpha(\sigma,k)}(H^{\sigma,k})(X_t)\, \mathrm{d}\, X_t + \frac{1}{2} \Delta\, U^{\alpha(\sigma,k)}(H^{\sigma,k})(X_t)\, \mathrm{d}\, t.$$

This allows us to equate the two continuous local martingales :

$$M_t - M_0 = \int_0^t F_s\, \mathrm{d}\, X_s, \qquad t \geqslant 0.$$

with

$$F_t := \sum_{k=0}^{n} \sum_{\sigma\in S_n} V_t^{\sigma,k}(X) e^{-\alpha(\sigma,k)t}\, \nabla U^{\alpha(\sigma,k)}(H^{\sigma,k})(X_t).$$

This shows that M can be written as a stochastic integral wrt X. So we have proven the theorem for any $\Phi \in \mathcal{C}$.

Step 2. General case. For a general $\Phi \in L^2$, by Lemma 3 we can choose a sequence $(\Phi_n \in \mathcal{C})_n$ such that $\Phi_n \to \Phi$ in L^2. Let $M_t^n := \mathbb{E}[\Phi_n | \mathcal{F}_t]$ and $\mathrm{M_t} := \mathbb{E}[\Phi | \mathcal{F}_\mathrm{t}]$. By Doob's maximal inequality (here we need càdlàg, so the right–continuity of the filtration), and passing to a subsequence if necessary, we can assume that $M^n \to M$ uniformly on any finite time interval. By Step 1, there exists a sequence of $L_{\mathcal{P}}^2$ processes $(F^n)_n$ such that

$$M_t^n = M_0^n + \int_0^t F_s^n \,\mathrm{d}\, X_s, \qquad t \geqslant 0.$$

Therefore

$$\mathbb{E}[(M_t^n - M_t^m)^2] = \mathbb{E}([M^n - M^m]_t) = \mathbb{E} \int_0^t |F_s^n - F_s^m|^2 \,\mathrm{d}\, s, \qquad t \geqslant 0,$$

which implies that F^n is Cauchy and converges in $L_{\mathcal{P}}^2 (\Omega \times [0, 1]; \mathbb{R}^d)$ to a predictable process F and that

$$M_t = M_0 + \int_0^t F_s \,\mathrm{d}\, X_s, \qquad t \geqslant 0. \qquad \square$$

Corollary 4. *All local martingales in a Brownian filtration are continuous.*

Exercise 1. Prove Corollary 4.

2 The variational properties of Girsanov transform

We assume that $(\Omega, \mathcal{F}, \mathcal{F}_\bullet, \mu)$ is the canonical d-dimensional Wiener space. That is $\Omega = C(\mathbb{R}_{\geqslant 0}; \mathbb{R}^d)$, $\mathcal{F} = \mathcal{B}(\Omega)$, X is the canonical process which under μ is a d-dimensional Brownian motion and the filtration $\mathcal{F}_\bullet = \mathcal{F}_\bullet^{X,\mu}$ is completed.

Definition 5. ***(Relative entropy)*** *Let Σ be a Polish space endowed with its Borel σ–field $\mathcal{B}(\Sigma)$, let $L^\infty(\Sigma)$, $\mathcal{M}_1(\Sigma)$ resp. the space of bounded measurable functions and the space of probability measures on Σ with the weak topology. Given two elements $\mu, \nu \in \mathcal{M}_1(\Sigma)$ the relative entropy of ν wrt. μ is defined as*

$$H(\nu|\mu) := \sup_{\varphi \in L^\infty(\Sigma)} (\nu(\varphi) - \log \mu(e^\varphi)). \tag{2}$$

The supremum in (2) can also be taken among continuous functions on Σ moreover we have the convex dual formula

$$\log \mu(e^\varphi) = \sup_{\nu \in \mathcal{M}_1(\Sigma)} [\nu(\varphi) - H(\nu|\mu)].$$

Lemma 6. *The function $\nu \mapsto H(\nu|\mu)$ is non–negative, lower semi–continuous, convex and moreover*

$$H(\nu|\mu) := \begin{cases} \int_\Sigma \log \frac{\mathrm{d}\nu}{\mathrm{d}\mu} \,\mathrm{d}\nu & \text{if } \nu \ll \mu, \\ +\infty & \text{otherwise.} \end{cases}$$

Proof. Since $\log(x) \leqslant x - 1$,

$$H(\nu|\mu) = -\int_{\mathcal{W}} \log\frac{\mathrm{d}\mu}{\mathrm{d}\nu}\,\mathrm{d}\nu \geqslant -\int_{\mathcal{W}} \frac{\mathrm{d}\mu}{\mathrm{d}\nu}\,\mathrm{d}\nu + 1 = 0.$$

Moreover

$$\lim_n H(\nu_n|\mu) \geqslant \lim_n (\nu_n(\varphi) - \log\mu(e^{\varphi})) = (\nu(\varphi) - \log\mu(e^{\varphi}))$$

and optimizing over φ we obtain $\lim_n H(\nu_n|\mu) \geqslant H(\nu|\mu)$. With a similar argument we prove convexity. □

We state some elementary but interesting relations between relative entropy and the Girsanov transform. In the following we call a *drift* a predictable function $u\colon \mathbb{R}_{\geqslant 0} \times \Omega \to \mathbb{R}^d$ such that

$$\|u\|_{\mathbb{H}}^2 := \int_0^{\infty} |u_s|^2 \mathrm{d}s$$

is finite μ–almost surely and denote $\mathcal{I}(u)_t := \int_0^t u_s \mathrm{d}s$, $t \geqslant 0$. Note that equivalently we can see u as a predictable function of X : $u_t(\omega) = u_t(X_\cdot(\omega))$. In order to stress the dependence on X we will sometimes write $u = u(X)$.

Lemma 7. *Let u be a drift and ν the law of $X + \mathcal{I}(u(X))$ under μ. Then*

$$H(\nu|\mu) \leqslant \frac{1}{2}\mathbb{E}_{\mu}\|u(X)\|_{\mathbb{H}}^2$$

Proof. Assume that $\|u\|_{\mathbb{H}}$ is almost surely bounded. By Novikov's criterion we can define the measure ρ with density

$$\tfrac{\mathrm{d}\rho}{\mathrm{d}\mu} = \mathcal{E}\left(-\int_0^{\cdot} u_s \mathrm{d}X_s\right)_{\infty}$$

with respect to μ. Under ρ the process $X + \mathcal{I}(u(X))$ is a Brownian motion, that is, it has law μ. Then

$$\mathbb{E}_{\nu}(f(X)) = \mathbb{E}_{\mu}(f(X + \mathcal{I}(u(X)))), \qquad \mathbb{E}_{\mu}(f(X)) = \mathbb{E}_{\rho}(f(X + \mathcal{I}(u(X))))$$

and using the definition of entropy we have

$$H(\nu|\mu) = \sup_{\varphi \in L^{\infty}(\Omega)} (\nu(\varphi) - \log\mu(e^{\varphi}))$$

$$= \sup_{\varphi \in L^{\infty}(\Omega)} \{\mathbb{E}_{\mu}[\varphi(X + \mathcal{I}(u(X)))] - \log\mathbb{E}_{\rho}[\exp\varphi(X + \mathcal{I}(u(X)))]\}$$

$$\leqslant \sup_{\varphi \in L^{\infty}(\Omega)} (\mu(\varphi) - \log\rho(e^{\varphi})) = H(\mu|\rho) = -\mathbb{E}_{\mu}\left[\log\mathcal{E}\left(-\int_0^{\cdot} u_s \mathrm{d}X_s\right)_{\infty}\right]$$

$$= \mathbb{E}_{\mu}\left[\int_0^{\infty} u_s \mathrm{d}X_s + \frac{1}{2}\|u\|_{\mathbb{H}}^2\right] = \frac{1}{2}\mathbb{E}_{\mu}[\|u\|_{\mathbb{H}}^2].$$

In the case of unbounded drifts we can introduce a sequence of appropriate stopping times $(\tau_n)_{n\geqslant 1}$ defined as $\tau_n := \inf(t\geqslant 0 \colon \|u\,\mathbb{I}_{[0,t]}\|_{\mathbb{H}}^2 \geqslant n)$ and stopped drifts $u_t^n = u_t\,\mathbb{I}_{t\leqslant\tau_n}$ so that $\|u^n\|_{\mathbb{H}}$ is bounded for all $n\geqslant 1$ and $\|u^n\|_{\mathbb{H}} \nearrow \|u\|_{\mathbb{H}}$ as $n\to\infty$. If $\|u\|_{\mathbb{H}} = +\infty$ with positive μ–probability we do not have anything to prove, so assume that $\|u\|_{\mathbb{H}} < +\infty$ μ–almost surely. Then $\tau_n\to\infty$ as $n\to\infty$ and $\mathcal{I}(u^n)\to\mathcal{I}(u)$ uniformly on compacts. As a result $X+\mathcal{I}(u^n)$ converges weakly to $X+\mathcal{I}(u)$ and, denoting ν^n and ν the respective laws, we have, by lower–semicontinuity of the entropy

$$H(\nu|\mu) \leqslant \liminf_n H(\nu_n|\mu) \leqslant \liminf_n \frac{1}{2}\mathbb{E}_\mu[\|u^n\|_{\mathbb{H}}^2] = \frac{1}{2}\mathbb{E}_\mu[\|u\|_{\mathbb{H}}^2]$$

(the last step by monotone convergence). □

2.1 Föllmer's drift

A natural question is in which condition equality is achieved in Lemma 7. We have the following Lemma, taken from Föllmer [3], which shows that any measure $\nu \ll \mu$ is associated with a specific drift.

Lemma 8. (Föllmer's drift) *Assume $\nu\ll\mu$. There exists a unique drift u such that $\|u\|_{\mathbb{H}}<\infty$ almost surely, $X-\mathcal{I}(u)$ is a ν–Brownian motion and*

$$H(\nu|\mu) = \frac{1}{2}\mathbb{E}_\nu[\|u\|_{\mathbb{H}}^2].$$

Proof. The density $Z = \mathrm{d}\nu/\mathrm{d}\mu$ defines a positive martingale via $Z_t := \mathbb{E}_\mu[Z|\mathcal{F}_t]$. Let $Z_t = \mathcal{E}(L)_t$. Let $(\tau_n)_n$ a reducing sequence for L (which may be just a càdlàg local martingale). By the martingale representation theorem we have $L^{\tau_n} = \int_0^t u^n\,\mathrm{d}X$, so in particular $u_t^n \mathbb{I}_{t<\tau_m} = u_t^m \mathbb{I}_{t<\tau_m}$ for all $m\leqslant n$ and $t\geqslant 0$ and then $L_t = \int_0^t u_s\,\mathrm{d}X_s$ is a continuous local martingale (since it is a stochastic integral) where $u_t = \lim_n u_t^n \mathbb{I}_{t<\tau_n}$. Now,

$$H(\nu|\mu) = \mathbb{E}_\nu\left[\log\frac{\mathrm{d}\nu}{\mathrm{d}\mu}\right] = \underbrace{\mathbb{E}_\nu\left[\log\frac{\mathrm{d}\nu}{\mathrm{d}\nu^n}\right]}_{H(\nu|\nu^n)\geqslant 0} + \mathbb{E}_\nu\left[\log\frac{\mathrm{d}\nu^n}{\mathrm{d}\mu}\right]$$

$$\geqslant \mathbb{E}_\nu\left[\log\frac{\mathrm{d}\nu^n}{\mathrm{d}\mu}\right] = \mathbb{E}_\nu\left[\int_0^{\tau_n} u_s \mathrm{d}X_s - \frac{1}{2}\int_0^{\tau_n}|u_s|^2\mathrm{d}s\right] = \frac{1}{2}\mathbb{E}_\nu\left[\int_0^{\tau_n}|u_s|^2\mathrm{d}s\right].$$

By monotone convergence we have $\mathbb{E}_\nu(\|u\|_{\mathbb{H}}^2) \leqslant 2H(\nu|\mu)$. Assume $H(\nu|\mu) < +\infty$. This implies that under ν the process $\tilde{L} = L - [L]$ is an L^2 bounded continuous martingale since $[\tilde{L}]_\infty = [L]_\infty = \|u\|_{\mathbb{H}}^2$ and

$$H(\nu|\mu) = \mathbb{E}_\nu\left[L_\infty - \frac{1}{2}[L]_\infty\right] = \mathbb{E}_\nu\left[\tilde{L}_\infty + \frac{1}{2}[L]_\infty\right] = \mathbb{E}_\nu\left[\tilde{L}_\infty + \frac{1}{2}[L]_\infty\right] = \frac{1}{2}\mathbb{E}_\nu(\|u\|_{\mathbb{H}}^2).$$

If $H(\nu|\mu) = +\infty$ by lower semicontinuity of the entropy we have $H(\nu_n|\mu) \to +\infty$ and since $\nu_n = \nu|_{\mathcal{F}_{\tau_n}}$ we have also $\mathbb{E}_\nu[\int_0^{\tau_n}|u_s|^2\mathrm{d}s] = \mathbb{E}_{\nu_n}[\int_0^{\tau_n}|u_s|^2\mathrm{d}s] = H(\nu_n|\mu) \to +\infty$ so by monotone convergence $\mathbb{E}_\nu[\int_0^\infty |u_s|^2\mathrm{d}s] = \lim_n \mathbb{E}_\nu[\int_0^{\tau_n}|u_s|^2\mathrm{d}s] = +\infty$ which verifies the formula also in this case.

For uniqueness just observe that if $\tilde{u}$ and u are two drifts satisfing the properties of the claim then $B = X - \mathcal{I}(u)$ and $\tilde{B} = X - \mathcal{I}(\tilde{u})$ are two ν–Brownian motions so $M = B - \tilde{B} = \mathcal{I}(\tilde{u} - u)$ is a continuous martingale. This implies that $[M]_\infty = [\mathcal{I}(\tilde{u} - u)]_\infty = 0$ and thus that $M_t = M_0 = 0$ for all t. That is $u_t(\omega) = \tilde{u}_t(\omega)$ for almost all t, ω. □

For a class of nice densities we have precise informations about the Föllmer drift.

Lemma 9. *Let ν be a probability measure which are absolutely continuous wrt μ with density Z such that $Z \in \mathcal{C}$ and $Z \geqslant \varepsilon$ for some $\varepsilon > 0$. Under ν the canonical process X is a strong solution of the SDE*

$$\mathrm{d}X_t = u_t(X)\mathrm{d}t + \mathrm{d}W_t \tag{3}$$

where W is a ν–Brownian motion and u a drift such that

$$|u_t(x) - u_t(y)| \leqslant L\|x - y\|_{\infty,[0,t]}, \qquad x, y \in C(\mathbb{R}_{\geqslant 0}; \mathbb{R}^d) \tag{4}$$

for some finite constant L. Moreover

$$H(\nu|\mu) = \frac{1}{2}\mathbb{E}_\nu \|u(X)\|^2.$$

We will call $\mathcal{S}_\mu$ the class of such measures.

Proof. Let $Z_t(X) := \mathbb{E}[Z|\mathcal{F}_t]$, by the proof of the Martingale Representation Theorem 1 we have that

$$Z_t(X) = 1 + \int_0^t F_s(X)\mathrm{d}X_s, \qquad t \geqslant 0,$$

where $Z(X)$ and $F(X)$ are, respectively, linear combination of functions of the form

$$\sum_{k=0}^{n} \sum_{\sigma \in S_n} V_t^{\sigma,k}(X) e^{-\alpha(\sigma,k)t}\, U^{\alpha(\sigma,k)}(H^{\sigma,k})(X_t), \qquad \sum_{k=0}^{n} \sum_{\sigma \in S_n} V_t^{\sigma,k}(X) e^{-\alpha(\sigma,k)t}\, \nabla U^{\alpha(\sigma,k)}(H^{\sigma,k})(X_t),$$

for various choices of $\alpha > 0$, $f \in C_0^\infty$. Moreover $Z_t(X) \geqslant \varepsilon$, so it is clear that if we let

$$u_t(x) := \frac{F_t(x)}{Z_t(x)}, \qquad x \in C(\mathbb{R}_{\geqslant 0}; \mathbb{R}^d)$$

we will satisfy (4) and moreover

$$Z_t(X) = 1 + \int_0^t Z_s(X) u_s(X)\mathrm{d}X_s.$$

This last relation implies that $Z(X) = \mathcal{E}(u(X))$. So by Girsanov theorem, the canonical process is a weak solution of the SDE (3). Due to condition (4) this SDE has the pathwise uniqueness property, so by the Yamada–Watanabe the canonical process X is a strong solution of (3) depending on the ν–Brownian motion W. Finally

$$H(\nu|\mu) = \mathbb{E}_\nu \log(Z) = \mathbb{E}_\nu\left[\int_0^\infty u_s(X)\mathrm{d}X_s - \frac{1}{2}\|u(X)\|^2\right] = \mathbb{E}_\nu\left[\int_0^\infty u_s(X)\mathrm{d}W_s + \frac{1}{2}\|u(X)\|^2\right]$$

if $\mathbb{E}_\nu \|u(X)\|^2 < +\infty$ the process $\int_0^\infty u_s(X) \mathrm{d}W_s$ is a L^2 martingale so its expectation vanish. A localization argument shows that the equality holds also in the case $\mathbb{E}_\nu \|u(X)\|^2 = +\infty$. $\square$

2.2 The Boué–Dupuis formula

The aim of this section is to prove the following variational caracterisation of expectations of positive functions over the Wiener space, due to Boué and Dupuis [1]. The proof is taken from Lehec [4].

Theorem 10. (Boué–Dupuis) *For any function $f:\Omega \to \mathbb{R}$ measurable and bounded from below we have*

$$\log \int_\Omega e^f \mathrm{d}\mu = \sup_u \mathbb{E}_\mu \left[f\left(X + \int_0^\cdot u_s(X) \mathrm{d}s \right) - \frac{1}{2} \|u(X)\|^2 \right]$$

where the supremum is taken over all the adapted drifts.

Lemma 11. *Let $f:\Omega \to \mathbb{R}$ be bounded from below. For every $\varepsilon > 0$ there exists $\nu \in \mathcal{S}_\mu$ such that*

$$\log \int_\Omega e^f \mathrm{d}\mu \leqslant \int_\Omega f \mathrm{d}\nu - H(\nu|\mu) + \varepsilon. \tag{5}$$

Proof. By monotone convergence it is enough to consider bounded functions f and that $\int_\Omega e^f \mathrm{d}\mu = 1$. Let $F = e^f$ and let ν be probability measure on Ω. Using

$$x \log(x) \leqslant |x-1| + |x-1|^2/2, \qquad x \geqslant 0$$

we get

$$H(\nu|\mu) - \int f \mathrm{d}\nu \leqslant \int \left| \frac{G}{F} - 1 \right| F \mathrm{d}\mu + \frac{1}{2} \int \left| \frac{G}{F} - 1 \right|^2 F \mathrm{d}\mu \leqslant \|F - G\|_{L^1(\mu)} + C_f \|F - G\|_{L^2(\mu)}^2$$

where G is the density of ν and C_f is some constant depending only on the lowe bound of f (recall that f is bounded from below). Given F and given the density of $\mathcal{C}$ in $L^2(\mu)$ we can find an element $G \in \mathcal{C}$ such that

$$\|F - G\|_{L^1(\mu)} + C_f \|F - G\|_{L^2(\mu)}^2 \leqslant \|F - G\|_{L^2(\mu)} + C_f \|F - G\|_{L^2(\mu)}^2 \leqslant \varepsilon$$

and for which $G \geqslant \delta$ for some $\delta > 0$. The proof is complete letting this G be the density of ν wrt μ and observing that $\nu \in \mathcal{S}_\mu$ by construction. $\square$

Proof. (of Theorem 10) Let u be a drift and ν the law of $X + \int_0^\cdot u_s(X) \mathrm{d}s$. By Lemma 7 and the variational caracterisation of the entropy (2) we have

$$\mathbb{E}_\mu \left[f\left(X + \int_0^\cdot u_s(X) \mathrm{d}s \right) - \frac{1}{2} \|u(X)\|^2 \right] \leqslant \int f \mathrm{d}\nu - H(\nu|\mu) \leqslant \log \int_\Omega e^f \mathrm{d}\mu.$$

On the other hand, given $\varepsilon > 0$ there exists a probability measure $\nu \in \mathcal{S}_\mu$ satisfying (5). Since $\nu \in \mathcal{S}_\mu$ Lemma 9 guarantee the existence of a strong drift z for which $\mathrm{d}X = z(X)\mathrm{d}t + \mathrm{d}W$ with a ν–Brownian motion W. Moreover X is a strong solution, so there exists a measurable map Φ such that $X = \Phi(W)$. As a consequence, letting $u(W) = z(\Phi(W))$ we have $\mathrm{d}X = u(W)\mathrm{d}t + \mathrm{d}W$. With this choice we can write

$$\int f\mathrm{d}\nu - H(\nu|\mu) = \mathbb{E}_\nu\left[f\left(W + \int_0^\cdot u_s(W)\mathrm{d}s\right)\right] - \frac{1}{2}\mathbb{E}_\nu[\|u(W)\|^2] = \mathbb{E}_\mu\left[f\left(X + \int_0^\cdot u_s(X)\mathrm{d}s\right) - \frac{1}{2}\|u(X)\|^2\right]$$

since the law of X under μ coincides with the law of W under ν. Then

$$\log\int_\Omega e^f\mathrm{d}\mu \leqslant \mathbb{E}_\mu\left[f\left(X + \int_0^\cdot u_s(X)\mathrm{d}s\right) - \frac{1}{2}\|u(X)\|^2\right] + \varepsilon$$

optimizing over u and letting $\varepsilon \to 0$ gives the claim. □

2.3 Some applications

A first consequence of Theorem 10 is a Gaussian bound on certain functionals of Brownian motion.

Corollary 12. *Let (E,d) some metric space and $f: \Omega \to E$ such that there exists an $e \in E$ for which*

$$d(f(x + \mathcal{I}(h)), e) \leqslant c(x)(g(x) + \|h\|_{\mathbb{H}}), \qquad h \in \mathbb{H}$$

for μ-a.e. $x \in \Omega$ where $c \in L^2(\mu)$ and $\mu(c\,g) < +\infty$. Then for all $\lambda > 0$

$$\mathbb{E}_\mu[\exp[\lambda d(f(X), e)]] \leqslant \exp(\lambda^2\mu(c^2)/2 + \lambda\mu(c\,g)).$$

In particular the random variable $d(f(X), e)$ has Gaussian tails.

Proof. By the Boué–Dupuis formula we have

$$\log\mathbb{E}_\mu[\exp[\lambda d(f(X), e)]] = \sup_u \mathbb{E}_\mu\left[\lambda d(f(X + \mathcal{I}(u)), e) - \frac{1}{2}\|u\|_{\mathbb{H}}^2\right]$$

By hypothesis on f:

$$\log\mathbb{E}_\mu[\exp[\lambda d(f(X), e)]] \leqslant \sup_u \mathbb{E}_\mu\left[\lambda c(X)(g(X) + \|u\|_{\mathbb{H}}) - \frac{1}{2}\|u\|_{\mathbb{H}}^2\right].$$

Now we can take an unrestricted sup over the r.h.s. with respect to all measurable functions $\theta: \Omega \to \mathbb{R}_+$ and obtain

$$\log\mathbb{E}_\mu[\exp[\lambda d(f(X), e)]] \leqslant \mathbb{E}_\mu[\lambda c(X)g(X)] + \sup_\theta \mathbb{E}_\mu\left[\lambda c(X)\theta(X) - \frac{1}{2}\theta(X)^2\right]$$

$$\leqslant \mathbb{E}_\mu[\lambda c(X)g(X)] + \frac{\lambda^2}{2}\mathbb{E}_\mu[c(X)^2] - \frac{1}{2}\sup_\theta \mathbb{E}_\mu[(\theta(X) - \lambda c(X))^2] \leqslant \lambda\mathbb{E}_\mu[c(X)g(X)] + \frac{\lambda^2}{2}\mathbb{E}_\mu[c(X)^2].$$

□

Another application of these considerations is the derivation of a transportation cost inequality of Talagrand in the case of Wiener measure.

Let ρ, ν be two probability measures on Ω and define

$$T_2(\nu, \rho) := \inf_{\pi} \left[\int_{\Omega \times \Omega} \left\| \frac{\mathrm{d}}{\mathrm{d}t}(\omega - \omega') \right\|_{\mathbb{H}}^2 \pi(\mathrm{d}\omega \times \mathrm{d}\omega') \right]^{1/2}$$

where the infimum is taken over all the probability measures π on $\Omega \times \Omega$ with fixed marginals ν, ρ, i.e. such that $\omega(\cdot \times \Omega) = \nu$ and $\omega(\Omega \times \cdot) = \rho$.

Theorem 13. *We have*

$$T_2(\nu, \mu) \leqslant [2\, H(\nu | \mu)]^{1/2}.$$

Proof. It is enough to consider the case $H(\nu|\mu) < +\infty$. Then Lemma 8 state the existence of a drift u and a Brownian motion B such that $X = B + \mathcal{I}(u)$ has law ν and $2\, H(\nu|\mu) = \mathbb{E}_\nu \|u\|_{\mathbb{H}}^2$. Let π be the law of (X, B). The first marginal of π is ν and the second is μ so

$$T_2(\nu, \mu)^2 \leqslant \mathbb{E}\left[\left\| \frac{\mathrm{d}}{\mathrm{d}t}(X - B) \right\|_{\mathbb{H}}^2 \right] = \mathbb{E}[\|u\|_{\mathbb{H}}^2] = 2\, H(\nu|\mu).$$

□

3 Small noise and large deviations

In this section we investigate the behaviour of probabilities of certain functionals of Brownian motion. Applications will be given to small noise limit of stochastic differential equations. Let us first introduce some general tools from Large Deviations theory. In the following $\mathcal{E}$ will be a Polish space (separable completely metrizable topological space).

Definition 14. *A function $I: \mathcal{E} \to [0, +\infty]$ is called a* (good) rate function *on $\mathcal{E}$ if the sets $I^{-1}[0, M] \subseteq \mathcal{E}$ are compacts for all $M < +\infty$.*

In particular a rate function is always lower semicontinuous, that is $\lim_{y \to x} I(y) \geqslant I(x)$, or equivalently that level sets $I^{-1}[0, M]$ are closed.

Definition 15. *Let I be a rate function on $\mathcal{E}$. A family $(Y^\varepsilon)_\varepsilon$ of random elements of $\mathcal{E}$ satisfies the Laplace principle on $\mathcal{E}$ with rate function I (and rate $1/\varepsilon$) if for all functions $h \in C_b(\mathcal{E})$ (continuous and bounded functions) we have*

$$-\lim_{\varepsilon \to 0} \varepsilon \log \mathbb{E}[\exp(-h(Y^\varepsilon)/\varepsilon)] = \inf_{x \in \mathcal{E}} [I(x) + h(x)]. \tag{6}$$

By general results this Laplace principle is equivalent in the Polish setting to exponential estimates of events for the family $(Y^\varepsilon)_\varepsilon$:

Definition 16. *A family* $(Y^\varepsilon)_\varepsilon$ *of random elements of* $\mathcal{E}$ *satisfies the Large Deviations principle on* $\mathcal{E}$ *with rate function* I *(and rate* $1/\varepsilon$*) if for any open set* $A\in\mathcal{E}$ *and closed set* $B\in\mathcal{E}$ *we have*

$$\liminf_{\varepsilon\to 0} \varepsilon\log\mathbb{P}(Y^\varepsilon\in A) \geqslant -\inf_{x\in A} I(x), \qquad \limsup_{\varepsilon\to 0}\varepsilon\log\mathbb{P}(Y^\varepsilon\in B)\leqslant -\inf_{x\in B} I(x).$$

Exercise 2. Prove the equivalence between the Laplace principle and the Large Deviations principle.

We will consider families of random variable $(Y^\varepsilon)_\varepsilon$ obtained from a fixed Brownian motion X via measurable mappings $\mathcal{G}^\varepsilon:\Omega\to\mathcal{E}$, $\varepsilon>0$.

Let $\mathbb{U}_M\subseteq L^2(\mathbb{R}_{\geqslant 0};\mathbb{R}^d)$ the subset of elements u such that $\|u\|_{\mathbb{H}}\leqslant M$ and let $\mathcal{U}_M\subseteq L^2_{\mathcal{P}}(\mathbb{R}_{\geqslant 0}\times\Omega;\mathbb{R}^d)$ the subset of predictable L^2 processes u such that $\|u\|_{\mathbb{H}}\leqslant M$ μ–almost surely. Note that $\mathbb{U}_M$ is a compact Polish space with respect to the weak–topology of $L^2(\mathbb{R}_{\geqslant 0};\mathbb{R}^d)$.

We will follow closely [2] and make the following general hypothesis on the family $(\mathcal{G}^\varepsilon)_\varepsilon$.

Hypothesis 17. *There exists measurable mapping* $\mathcal{G}^0:\Omega\to\mathcal{E}$ *such that the following holds*

i. for every $M<\infty$ *and any family* $(u^\varepsilon)_\varepsilon\subseteq\mathcal{U}_M$ *such that* u^ε *converges in distribution (as random elements of* $\mathbb{U}_M$*) to* u *we have that* $\mathcal{G}^\varepsilon(X+\varepsilon^{-1/2}\mathcal{I}(u^\varepsilon))\to\mathcal{G}^0(\mathcal{I}(u))$ *in distribution (as random elements of* $\mathcal{E}$*);*

ii. for every $M<\infty$ *the set* $\Gamma_M:=\{\mathcal{G}^0(\mathcal{I}(u)):u\in\mathbb{U}_M\}$ *is a compact subset of* $\mathcal{E}$*.*

For each $x\in\mathcal{E}$ define

$$I(x):=\frac{1}{2}\inf_{u\in\Gamma(x)}\|u\|^2_{\mathbb{H}} \tag{7}$$

where the inf is taken over the set $\Gamma(x)\subseteq\mathbb{H}$ of all $u\in\mathbb{H}$ such that $x=\mathcal{G}^0(\mathcal{I}(u))$ and is taken to be $+\infty$ if this set is empty. Under Hypothesis 17 the function I is a rate function on $\mathcal{E}$.

Theorem 18. *Under Hypothesis 17 the family* $(Y^\varepsilon=\mathcal{G}^\varepsilon(X))_\varepsilon$ *satisfies a Laplace principle with rate function* I *as defined in (7).*

Proof. We need to show that (6) holds for all $h\in C_b(\mathcal{E})$.

Step 1: Lower bound. By the Boué–Dupuis formula

$$\begin{aligned}-\varepsilon\log\mathbb{E}_\mu[\exp(-h(Y^\varepsilon)/\varepsilon)] &= \inf_u\mathbb{E}_\mu\left[\frac{1}{2}\|\varepsilon^{1/2}u\|^2_{\mathbb{H}}+h(\mathcal{G}^\varepsilon(X+\mathcal{I}(u)))\right]\\ &=\inf_u\mathbb{E}_\mu\left[\frac{1}{2}\|u\|^2_{\mathbb{H}}+h(\mathcal{G}^\varepsilon(X+\varepsilon^{-1/2}\mathcal{I}(u)))\right].\end{aligned}$$

Fix $\delta>0$. For every $\varepsilon>0$ there exists u^ε such that

$$-\varepsilon\log\mathbb{E}_\mu[\exp(-h(Y^\varepsilon)/\varepsilon)]\geqslant\mathbb{E}_\mu\left[\frac{1}{2}\|u^\varepsilon\|^2_{\mathbb{H}}+h(\mathcal{G}^\varepsilon(X+\varepsilon^{-1/2}\mathcal{I}(u^\varepsilon)))\right]-\delta.$$

Moreover

$$\mathbb{E}_\mu\left[\frac{1}{2}\|u^\varepsilon\|_{\mathbb{H}}^2\right] \leqslant \|h\|_\infty - \varepsilon\log\mathbb{E}[\exp(-h(X^\varepsilon)/\varepsilon)] + \delta \leqslant 2\|h\|_\infty + \delta.$$

Modulo taking N large enough we can replace u^ε by the stopped process $u_t^{\varepsilon,N} = u_t^\varepsilon \mathbb{I}_{t\leqslant\tau_{\varepsilon,N}}$ where $\tau_{\varepsilon,N} = \inf\{t\geqslant 0 : \|u^\varepsilon \mathbb{I}_{[0,t]}\|_{\mathbb{H}} > N\}$. Indeed observe that

$$\mathbb{P}(u^\varepsilon \neq u^{\varepsilon,N}) \leqslant \mathbb{P}(\|u^\varepsilon\|_{\mathbb{H}} > N) \leqslant \frac{\mathbb{E}_\mu[\|u^\varepsilon\|_{\mathbb{H}}^2]}{N} \leqslant \frac{4\|h\|_\infty + 2\delta}{N}$$

uniformly in ε. This implies that we can choose N large enough uniformly in ε so that

$$|\mathbb{E}_\mu[h(\mathcal{G}^\varepsilon(X + \varepsilon^{-1/2}\mathcal{I}(u^\varepsilon)))] - \mathbb{E}_\mu[h(\mathcal{G}^\varepsilon(X + \varepsilon^{-1/2}\mathcal{I}(u^{\varepsilon,N})))]| \leqslant \|h\|_\infty \mathbb{P}(u^\varepsilon \neq u^{\varepsilon,N}) \leqslant \delta$$

and of course $\mathbb{E}_\mu[\|u^\varepsilon\|_{\mathbb{H}}^2] \geqslant \mathbb{E}_\mu[\|u^{\varepsilon,N}\|_{\mathbb{H}}^2]$, so we have

$$-\varepsilon\log\mathbb{E}_\mu[\exp(-h(Y^\varepsilon)/\varepsilon)] \geqslant \mathbb{E}_\mu\left[\frac{1}{2}\|u^{\varepsilon,N}\|_{\mathbb{H}}^2 + h(\mathcal{G}^\varepsilon(X + \varepsilon^{-1/2}\mathcal{I}(u^{\varepsilon,N})))\right] - 2\delta.$$

In this situation $\|u^{\varepsilon,N}\|_{\mathbb{H}} \leqslant N$ almost surely for every $\varepsilon > 0$ so we can extract a weakly converging subsequence (still denoted $u^{\varepsilon,N}$) and let $u \in \mathcal{U}_N$ be its limit. Using Hypothesis 17 we have

$$\liminf_{\varepsilon\to 0} -\varepsilon\log\mathbb{E}_\mu[\exp(-h(Y^\varepsilon)/\varepsilon)] \geqslant \mathbb{E}_\mu\left[\frac{1}{2}\|u\|_{\mathbb{H}}^2 + h(\mathcal{G}^0(\mathcal{I}(u)))\right] - 2\delta$$

$$\geqslant \inf_{x\in\mathcal{E}} \inf_{v\in\Gamma(x)} \left[\frac{1}{2}\|\nu\|_{\mathbb{H}}^2 + h(x)\right] - 2\delta \geqslant \inf_{x\in\mathcal{E}} [I(x) + h(x)] - 2\delta.$$

Since δ is arbitrary this completes the proof of the lower bound.

Step 2: Upper bound. Using that h is bounded we have $I(h) := \inf_{x\in\mathcal{E}} [I(x) + h(x)] < +\infty$. Let $\delta > 0$ and choose $x_0 \in \mathcal{E}$ such that $I(x_0) + h(x_0) \leqslant I(h) + \delta/2$. Moreover choose $v \in L^2(\mathbb{R}_{\geqslant 0}; \mathbb{R}^d)$ such that $\|v\|_{\mathbb{H}}^2/2 \leqslant I(x_0) + \delta/2$ and $x_0 = \mathcal{G}(\mathcal{I}(v))$. By the Boué–Dupuis formula

$$\limsup_{\varepsilon\to 0} -\varepsilon\log\mathbb{E}_\mu[\exp(-h(Y^\varepsilon)/\varepsilon)] = \limsup_{\varepsilon\to 0} \inf_u \mathbb{E}_\mu\left[\frac{1}{2}\|u\|_{\mathbb{H}}^2 + h(\mathcal{G}^\varepsilon(X + \varepsilon^{-1/2}\mathcal{I}(u)))\right]$$

$$\leqslant \limsup_{\varepsilon\to 0} \mathbb{E}_\mu\left[\frac{1}{2}\|v\|_{\mathbb{H}}^2 + h(\mathcal{G}^\varepsilon(X + \varepsilon^{-1/2}\mathcal{I}(v)))\right]$$

$$= \limsup_{\varepsilon\to 0} \left\{\frac{1}{2}\|v\|_{\mathbb{H}}^2 + \mathbb{E}_\mu[h(\mathcal{G}^\varepsilon(X + \varepsilon^{-1/2}\mathcal{I}(v)))]\right\}.$$

By assumption $\mathcal{G}^\varepsilon(X + \varepsilon^{-1/2}\mathcal{I}(v))$ weakly converges to $\mathcal{G}(\mathcal{I}(v)) = x_0$ so

$$\limsup_{\varepsilon\to 0} -\varepsilon\log\mathbb{E}_\mu[\exp(-h(Y^\varepsilon)/\varepsilon)] \leqslant \frac{1}{2}\|v\|_{\mathbb{H}}^2 + h(x_0) \leqslant I(x_0) + h(x_0) + \delta/2 = I(h) + \delta.$$

Since δ is arbitrary the proof is complete. □

Example 19. The simplest situation is when $\mathcal{E}=\Omega$ and $Y^{\varepsilon}=\varepsilon^{1/2}X$. In this case $\mathcal{G}^0(\mathcal{I}(u))=\mathcal{I}(u)$ and we leave to the readed to check that Hypothesis 17 holds. Then we have established that

$$-\lim_{\varepsilon\to 0}\varepsilon\log\mathbb{E}_{\mu}[\exp(-h(\varepsilon^{1/2}X)/\varepsilon)]=\inf_{u\in L^2(\mathbb{R}_{\geqslant 0};\mathbb{R}^d)}\left[\frac{1}{2}\|u\|_{\mathbb{H}}^2+h(\mathcal{I}(u))\right]. \tag{8}$$

which is the Laplace version of Schilder's theorem about large deviations of Brownian motion.

Bibliography

[1] Michelle Boué and Paul Dupuis. A variational representation for certain functionals of Brownian motion. *The Annals of Probability*, 26(4):1641–1659, oct 1998.

[2] Amarjit Budhiraja and Paul Dupuis. A variational representation for positive functionals of infinite dimensional Brownian motion. *Probability and Mathematical Statistics*, 20(1, Acta Univ. Wratislav. No. 2246):39–61, 2000.

[3] H. Föllmer. An entropy approach to the time reversal of diffusion processes. In Prof M. Metivier and Prof E. Pardoux, editors, *Stochastic Differential Systems Filtering and Control*, number 69 in Lecture Notes in Control and Information Sciences, pages 156–163. Springer Berlin Heidelberg, 1985. DOI: 10.1007/BFb0005070.

[4] Joseph Lehec. Representation formula for the entropy and functional inequalities. *Annales de l'Institut Henri Poincaré Probabilités et Statistiques*, 49(3):885–899, 2013.

[5] P. A. Meyer. *Processus de Markov*. Lecture Notes in Mathematics 26. Springer-Verlag, 1967.

[6] Zhongmin Qian and Jiangang Ying. Martingale representations for diffusion processes and backward stochastic differential equations. *ArXiv:0910.4911 [math]*, oct 2009. ArXiv: 0910.4911.