Stochastic Analysis – Course note 4

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1 Regularity of stochastic processes

We analyse here general conditions under which a family of random variables defined on $\mathbb{R}^k$ enjoys a *continuous version*. Sufficient conditions are given by the theorem below, essentially due to Kolmogorov. However we will prove it by showing first a relation between the Hölder norm and an integral norm which is related to the theory of Besov spaces. Kolmogorov theorem can be seen as an application of an embedding of Hölder spaces into certain Besov spaces. From this point of view one can obtain also useful informations about integrability of norms which do not enter the original formulation of Kolmogorov theorem.

Theorem 1. (Kolmogorov) *Let $(X_t)_{t\in\mathbb{R}^k}$ a family of random variables indexed by a k–dimensional parameter $t\in\mathbb{R}^k$. Assume that for some $p>0,\alpha>k/p$ we have*

$$\mathbb{E}[|X_t-X_s|^p]\lesssim|t-s|^{\alpha p}\qquad t,s\in\mathbb{R}^k \tag{1}$$

then there exists a random variable $\tilde{X}$ with values in $C(\mathbb{R}^k;\mathbb{R})$ such that $\mathbb{P}(X_t=\tilde{X}_t)=1$ for all $t\in\mathbb{R}^k$ and such that for all $L>0$, and $\gamma<\alpha-k/p$,

$$\sup_{t,s\in[-L,L]^k}\frac{|\tilde{X}(\omega)_t-\tilde{X}(\omega)_s|}{|t-s|^\gamma}\leq C_{L,\gamma}(\omega)<\infty,$$

for $\mathbb{P}$–almost all $\omega\in\Omega$.

The proof of this theorem relies on a lemma due to Garcia, Rodemich et Rumsey which allows the point-wise control on the regularity of a continuous function via an integral quantity. This formulation of the GRR lemma is taken from [4] with a slight modification. See also [3].

In the lemma we consider a nice metric space (Λ,d) endowed with a measure m (on the Borel sets of Λ). Denote $B(x,r)=\{y\in\Lambda:d(x,y)\leq r\}$ the ball of radius r centered in $x\in\Lambda$ and with $\sigma(r)=\inf_{x\in\Lambda}m(B(x,r))$ the smallest volume of a ball of radius r according to m.

We need to assume that $\sigma(r)>0$ for all $r>0$ and that if we denote

$$\overline{f}(A):=\int_A f(t)m(\mathrm{d}t)/m(A)$$

the mean of f on the Borel set A we have $\bar{f}(B(x,r))\to f(x)$ as $r\to0$ for all x and continuous function f.

We will also fix a function $\Psi:\mathbb{R}_{\geqslant0}\to\mathbb{R}_{\geqslant0}$ positive, increasing and convex and such that $\Psi(0)=0$ and denote with Ψ^{-1} its inverse : a positive, increasing and concave function.

To fix ideas one can think to $\Lambda=[0,1]$ with Euclidean distance $d(t,s)=|t-s|$, m Lebesgue measure, in this case $\sigma(r)=2r$. And take $\Psi(x)=x^p$.

Lemma 2. *[Garcia-Rodemich-Rumsey] Let $f:\Lambda\to\mathbb{R}$ a continuous function on (Λ,d). Let*

$$U=\int\int_{\Lambda\times\Lambda}\Psi\left(\frac{|f(t)-f(s)|}{d(t,s)}\right)m(\mathrm{d}t)m(\mathrm{d}s).$$

Then

$$|f(t)-f(s)|\leq 18\int_0^{d(t,s)/2}\Psi^{-1}\left(\frac{U}{\sigma(r)^2}\right)d\,r.$$

Proof. For Borel sets $A,B\subseteq\Lambda$ we have

$$\bar{f}(A)-\bar{f}(B)=\int\int_{A\times B}(f(t)-f(s))\frac{m(\mathrm{d}t)m(\mathrm{d}s)}{m(A)m(B)}$$

$$=\Psi^{-1}\left[\Psi\left(\int\int_{A\times B}d(t,s)\frac{f(t)-f(s)}{d(t,s)}\frac{m(\mathrm{d}t)m(\mathrm{d}s)}{m(A)m(B)}\right)\right]$$

Letting $d(A,B)=\sup_{t\in A,s\in B}d(t,s)$ we can estimate the difference of the means as

$$|\bar{f}(A)-\bar{f}(B)|\leq d(A,B)\Psi^{-1}\left[\Psi\left(\int\int_{A\times B}\frac{f(t)-f(s)}{d(t,s)}\frac{m(\mathrm{d}t)m(\mathrm{d}s)}{m(A)m(B)}\right)\right],$$

and by Jensens' inequality and convexity of Ψ,

$$\leq d(A,B)\Psi^{-1}\left[\int\int_{A\times B}\Psi\left(\frac{f(t)-f(s)}{d(t,s)}\right)\frac{m(\mathrm{d}t)m(\mathrm{d}s)}{m(A)m(B)}\right].$$

Which is boldly estimated as

$$|\bar{f}(A)-\bar{f}(B)|\leq d(A,B)\Psi^{-1}\left(\frac{U}{m(A)m(B)}\right). \tag{2}$$

Now let $\bar{f}(t,r):=\bar{f}(B(t,r))$ and $\lambda_k=d(t,s)/2^{-k}$ for $k=0,1,\ldots$. Then

$$|\bar{f}(t,\lambda_n)-\bar{f}(t,\lambda_0)|=\left|\sum_{k=0}^{n-1}(\bar{f}(t,\lambda_{k+1})-\bar{f}(t,\lambda_k))\right|\leqslant\sum_{k=0}^{\infty}|\bar{f}(t,\lambda_{k+1})-\bar{f}(t,\lambda_k)|$$
$$\leqslant\sum_{k=0}^{\infty}(\lambda_{k+1}+\lambda_k)\Psi^{-1}\left(\frac{U}{\sigma(\lambda_{k+1})\sigma(\lambda_k)}\right)$$

where we used eq. (2) noting that $d(B(t,a),B(t,b))=a+b$. We want to estimate this series via an integral, easier to manipulate. So we note that

$$\lambda_{k+1}+\lambda_k=32^{-k-1}d(t,s)=6(\lambda_{k+1}-\lambda_{k+2})$$

and that $\sigma(\lambda_k)\geqslant\sigma(\lambda_{k+1})\geqslant\sigma(r)$ for all $r\leqslant\lambda_{k+1}$. Then

$$|\bar{f}(t,\lambda_n)-\bar{f}(t,\lambda_0)|\leqslant 6\sum_{k=0}^{\infty}\int_{\lambda_{k+2}}^{\lambda_{k+1}}\Psi^{-1}\left(\frac{U}{\sigma(r)^2}\right)\mathrm{d}r=6\int_0^{d(t,s)/2}\Psi^{-1}\left(\frac{U}{\sigma(r)^2}\right)\mathrm{d}r$$

Now it is enough to take the limit $n\to\infty$ on the r.h.s. and use the continuity of f to conclude that the same integral gives a upper bound on $|\bar{f}(t,\lambda_n)-\bar{f}(t,\lambda_0)|$. Using again eq. (2) we have

$$|\bar{f}(t,\lambda_0)-\bar{f}(s,\lambda_0)|\leqslant 3\lambda_0\Psi^{-1}\left(\frac{U}{\sigma(\lambda_0)^2}\right)\leqslant 6\int_0^{d(t,s)/2}\Psi^{-1}\left(\frac{U}{\sigma(r)^2}\right)\mathrm{d}r$$

from which we obtain easily the claim. □

Let us now prove now Theorem 1.

Proof. Fix $L>0$. For every integer $N>0$ we consider the finite set $\Lambda_N=(\mathbb{Z}/2^N)^k\cap[-L,L]^k\subseteq\mathbb{R}^k$ and apply Lemma 2 to the function X_t defined on Λ_N and so trivially continuous. The measure m_N is the normalized counting measure on Λ_N, $d(t,s)=|t-s|^\beta$ and $\Psi(x)=x^p$. We can show aht there exists an constant $c>0$ such tha $\sigma(r)\geqslant cr^{k/\beta}$ for all $r>0$ uniformly in N. A direct computation gives

$$|X_t-X_s|\leqslant CU_N^{1/p}d(t,s)^{1-2k/\beta p}.$$

So

$$\mathbb{E}\left(\sup_{t,s\in\Lambda_N}\frac{|X_t-X_s|}{d(t,s)^{1-2k/\beta p}}\right)^p\leqslant C^p\int\int_{\Lambda_N\times\Lambda_N}\mathbb{E}\left(\frac{|X_t-X_s|}{|t-s|^\alpha}\right)^p\mathbb{I}_{t\neq s}\frac{m_N(\mathrm{d}t)m_N(\mathrm{d}s)}{|t-s|^{p(\beta-\alpha)}}$$

where we applied Fubini-Tonelli to exchange the integral with the expectation. This gives in turn

$$\mathbb{E}(Z_N^p)\leqslant C^p\sup_{t,s\in[-L,L]^k}\mathbb{E}\left(\frac{|X_t-X_s|}{|t-s|^\alpha}\right)^p\int\int_{\Lambda_N\times\Lambda_N}\mathbb{I}_{t\neq s}\frac{m_N(\mathrm{d}t)m_N(\mathrm{d}s)}{|t-s|^{p(\beta-\alpha)}}$$

where

$$Z_N:=\sup_{t,s\in\Lambda_N}\frac{|X_t-X_s|}{|t-s|^{\beta-2k/p}}.$$

The double integral is uniformly bounded in N if $p(\beta-\alpha)<k$ and in this way we obtain the uniform integrability of the random variables $(Z_N)_N$. Moreover $\Lambda_N\subseteq\Lambda_{N+1}$ and as a consequence the family $(Z_N)_N$ is increasing. By monotone convergence we have that

$$Z_\infty:=\sup_N Z_N=\sup_{t,s\in\Lambda_0}\frac{|X_t-X_s|}{|t-s|^{\beta-2k/p}}$$

is almost surely finite where $\Lambda_0=\cup_N\Lambda_N$ is a dense countable set in $[-L,L]^k$. Thanks to the condition $\alpha>k/p$ we can choose $\beta<\alpha+k/p$ such that $\beta-2k/p>0$. This implies that, almost surely, the random function $X:\Lambda_0\to\mathbb{R}$ is Hölder continuous of index $\gamma=\beta-2k/p<\alpha-k/p$ and it admits a continuous extension to all $[-L,L]^k$ which we will denote by $\tilde{X}$. It is now easy to see that condition (1) implies the continuity in probability of X and then that $\mathbb{P}(X_t=\tilde{X}_t)=1$ for all $t\in[-L,L]^k$. Being L arbitrary, we can build a consistent family of such continuous versions of X for an increasing sequence of values of L and then obtain a continous version of X over all $\mathbb{R}^k$. □

Example 3. If B is a standard one–dimensional Brownian motion we have

$$\mathbb{E}|B_t-B_s|^p=C_p|t-s|^{p/2}.$$

Then, after Theorem 1, there exists a version of B which is Hölder continuous for all $\gamma<1/2$. It is easy to see that the Hölder index cannot be $1/2$. Indeed

$$\mathbb{P}\left(\sup_{0<s<t<1}\frac{|B_t-B_s|}{|t-s|^{1/2}}=+\infty\right)=1. \tag{3}$$

Let us show this. We start by giving a lower bound on the Hölder norm. For all $n>0$, we consider the partition $\{t_k=k/n: k=0,\ldots,n\}\subseteq[0,1]$ and observe that

$$\sup_{0<s<t<1}\frac{|B_t-B_s|}{|t-s|^{1/2}}\geqslant\sup_{k=0,\ldots,n-1}A_k$$

where $A_k = |B_{t_{k+1}} - B_{t_k}| / |t_{k+1} - t_k|^{1/2}$. The r.v.s $\{A_k : k = 0, \ldots, n-1\}$ are an iid family with standard Gaussian law, so

$$\mathbb{P}\left(\sup_{0<s<t<1} \frac{|B_t - B_s|}{|t-s|^{1/2}} \geqslant L \right) \geqslant \mathbb{P}\left(\sup_{k=0,\ldots,n-1} A_k \geqslant L \right) = 1 - \mathbb{P}(A_1 < L)^n \to 1 \quad \text{for } n \to \infty.$$

Being L arbitrary, we obtain eq. (3).

Exercise 1. Apply Lemma 2 to Brownian motion with the function $\Psi(x) = e^{\lambda x^2} - 1$. What estimation this gives for $\rho(\delta) = \sup_{t,s:|t-s| \leqslant \delta} |B_t - B_s|$?

2 Regularity of SDEs and stochastic flows

Let $(V_k : \mathbb{R}_+ \times \mathbb{R}^d \to \mathbb{R}^d)_{k=0,\ldots,m}$ a family of time–dependent vectorfields on $\mathbb{R}^d$ and for $s > 0$ and $x \in \mathbb{R}^d$ let $t \mapsto \xi_{st}(x)$ the solution of the SDE

$$\xi_{st}(x) = x + \sum_{k=0}^{m} \int_s^t V_k(r, \xi_{sr}(x))\, \mathrm{d}B_r^k \tag{4}$$

where $(B_\cdot^k)_{k=1,\ldots,m}$ is a family of one–dimensional Brownian motions and where setting $B_t^0 = t$ gives a compact way to include a deterministic drift. We will assume that the vectorfields $(V_k)_k$ are globally Lipshitz and that it exists a constant M such that

$$|V_k(t,x) - V_k(t,y)| \leqslant M|x-y| \quad \text{and that} \quad |V_k(t,x)| \leqslant M(1+|x|)$$

uniformly in $t \geqslant 0$. In these conditions the SDE (4) admits a unique strong solution adapted to the filtration $\mathcal{F}_{s\bullet} = (\mathcal{F}_{st} = \sigma(B_u - B_s : u \geqslant s))_{t \geqslant s}$ generated by the increments of the Brownian motion after time s. By construction the process $t \mapsto \xi_{st}(x)$ is almost surely continuous for fixed s, x.

We are going to investigate the pathwise regularity of the process $(s,t,x) \mapsto \xi_{st}(x)$ with respect to all the variables (s,t,x) jointly. Out basic tool will be the Kolmogorov/Garcia-Rodemich-Rumsey Lemma, Itô formula and the Burkholder-Davis-Gundy inequalities. A very nice reference on the subject of stochastic flows are the lecture notes and the book of Kunita [2, 1].

The main result will be the following:

Theorem 4. *There exists a random function $(s, t, x) \mapsto \xi_{st}(x)$ which is Hölder continuous joinlty in s, t, x of exponents α, α, β for all $\alpha < 1/2$ and $\beta < 1$ and for which eq. (4) and the flow property $\xi_{ut}(\xi_{su}(x)) = \xi_{st}(x)$ are verified for all s, t, x almost surely.*

Proof. The proof is a direct consequence of Theorem 1 and of the estimation

$$\mathbb{E}|\xi_{st}(x) - \xi_{s't'}(x')|^p \lesssim |x-x'|^p + (1+|x|+|x'|)(|s-s'|^{p/2} + |t-t'|^{p/2}) \tag{5}$$

which we will prove in Theorem 8. Indeed, given a compact $K \subseteq \mathbb{R}^d$ and $T > 0$, the bound (5) is sufficient to apply Theorem 1 from which we obtain a continuous version in $(s,t,x) \in [0,T]^2 \times K$ of $\xi_{st}(x)$ such that

$$|\xi_{st}(x) - \xi_{s't'}(x')| \leqslant C_{K,T,p,\alpha,\beta}(\omega)(|t-t'|^\alpha + |s-s'|^\alpha + |x-x'|^\beta)$$

uniformly in s, t, x for all $\alpha < 1/2$ and $\beta < 1$. It is then easy to show the continuity of the stochastic integral in eq. (4) and then deduce that the SDE is satisfied for all s, t, x almost surely (the negligible set does not depend on t, s, x). The flow property follows from the regularity of $\xi_{st}(x)$ and Corollary 7. □

Remark 5. A map $(s,t,x)\mapsto\phi_{st}(x)$ satisfying $\phi_{ut}\circ\phi_{su}=\phi_{st}$ for all $s<u<t$ is called a *flow*. We proved that SDEs with regular coefficients give rise to a map ξ which is a *stochastic flow*.

Lemma 6. *For all $p\in\mathbb{R}$, $T>0$ and $\varepsilon>0$ we have*

$$\mathbb{E}(\varepsilon+|\xi_{st}(x)|^2)^p\leqslant C_{\varepsilon,p,T}(\varepsilon+|x|^2)^p$$

$$\mathbb{E}(\varepsilon+|\xi_{st}(x)-\xi_{st}(y)|^2)^p\leqslant C_{p,T}(\varepsilon+|x-y|^2)^p$$

for all $0\leqslant s\leqslant t\leqslant T$. The constant for the second inequality is uniform in ε.

Proof. Let $f(x):=(\varepsilon+|x|^2)$ and $F(x):=f(x)^p$. An easy computation gives

$$\nabla_i F(x)=2f(x)^{p-1}x \quad \nabla^2_{ij}F(x)=2pf(x)^{p-2}(f(x)\delta_{ij}+2(p-1)x_ix_j),\qquad i,j=1,\ldots,d$$

and if we denote $Z_t:=\xi_{st}(x)$ then, by Itô formula applied to the semimartingale $F(Z_t)$ we have

$$F(Z_t)=F(Z_s)+\sum_{i=1}^d\int_s^t\nabla_iF(Z_r)\mathrm{d}Z_r^i+\frac{1}{2}\sum_{i,j=1}^d\int_s^t\nabla^2_{ij}F(Z_t)\,\mathrm{d}\langle Z^i,Z^j\rangle_r$$

where

$$\mathrm{d}Z_r^i=\mathrm{d}\xi^i_{sr}(x)=\sum_{k=0}^m V_k^i(r,\xi_{sr}(x))\,\mathrm{d}B_r^k,$$

$$\mathrm{d}\langle Z^i,Z^j\rangle_r=\sum_{k,l=0}^m V_k^i(r,\xi_{sr}(x))V_l^j(r,\xi_{sr}(x))\,\mathrm{d}\langle B^k,B^l\rangle_r=\sum_{k=1}^m V_k^i(r,\xi_{sr}(x))V_k^j(r,\xi_{sr}(x))\,\mathrm{d}t$$

since $\langle B^k,B^l\rangle_t=t$ if $k=l=1,\ldots,m$ and 0 otherwise (in particular if $k=0$ or $l=0$, since $B_t^0=t$). Then

$$\begin{aligned}F(Z_t)=F(Z_s)&+\sum_{i=1}^d\sum_{k=0}^m\int_s^t\nabla_iF(Z_r)V_k^i(r,Z_r)\,\mathrm{d}B_r^k\\&+\frac{1}{2}\sum_{i,j=1}^d\sum_{k=1}^m\int_s^t\nabla^2_{ij}F(Z_t)V_k^i(r,Z_r)V_k^j(r,Z_r)\,\mathrm{d}r.\end{aligned}$$

Let us take the expectation of this last quantity: the stochastic integral vanishes (easy to see) and $Z_s=\xi_{ss}(x)=x$ a.s., so

$$\mathbb{E}F(Z_t)=F(x)+\frac{1}{2}\sum_{i,j=1}^d\sum_{k=1}^m\int_s^t\mathbb{E}[\nabla^2_{ij}F(Z_t)V_k^i(r,Z_r)V_k^j(r,Z_r)]\,\mathrm{d}r$$

To stimate the quantity in the integral we note that, by assumption $|V_k(r,x)|\leqslant M(1+|x|)\leqslant C_\varepsilon\sqrt{f(x)}$ (where the constant C_ε depends on ε). So

$$|\nabla^2_{ij}F(Z_t)V_k^i(r,Z_r)V_k^j(r,Z_r)|\leqslant C_\varepsilon F(Z_r)$$

and

$$\mathbb{E}F(Z_t)\leqslant F(x)+C_\varepsilon\int_s^t\mathbb{E}F(Z_r)\,\mathrm{d}r.$$

By Gronwall inequality we can conclude and obtain the first bound in our statement. For the second we proceed similarly. This time however we let $Z_t := \xi_{st}(x) - \xi_{st}(y)$. The process Z_t is still a semimartingale such that

$$\mathrm{d}Z_t = \sum_{k=0}^{m} [V_k^i(r, \xi_{sr}(x)) - V_k^i(r, \xi_{sr}(y))] \mathrm{d}B_r^k,$$

$$\mathrm{d}\langle Z^i, Z^j \rangle_r = \sum_{k=1}^{m} [V_k^i(r, \xi_{sr}(x)) - V_k^i(r, \xi_{sr}(y))][V_k^j(r, \xi_{sr}(x)) - V_k^j(r, \xi_{sr}(y))] \,\mathrm{d}t.$$

This time we have that

$$|V_k^i(r, \xi_{sr}(x)) - V_k^i(r, \xi_{sr}(y))| \leqslant M |\xi_{sr}(x) - \xi_{sr}(y)| \leqslant M f(Z_r)^{1/2}$$

independently of ε. Again by Gronwall we can conclude. □

Exercise 2. Show that we have $\mathbb{E}\left[\sup_{t \in [s,T]} (\varepsilon + |\xi_{st}(x) - \xi_{st}(y)|^2)^p\right] \leqslant C_{p,T} (\varepsilon + |x - y|^2)^p$.

Corollary 7. *For all $0 \leqslant s \leqslant u \leqslant t \leqslant T$ we have almost surely $\xi_{ut}(\xi_{su}(x)) = \xi_{st}(x)$ for all $x \in \mathbb{R}^d$.*

Proof. Lemma 6 and the Kolmorov theorem imply that for all fixed s, t, the map $x \mapsto \xi_{st}(x)$ is almost surely continuous (the exceptional set depending on s, t). Moreover it is easy to see that we can choose the family of random variables

$$x \mapsto \sum_{k=0}^{m} \int_s^t V_k(r, \xi_{sr}(x)) \,\mathrm{d}B_r^k$$

continuous in x. Indeed

$$\begin{aligned}
&\mathbb{E}\left[\int_s^t V_k(r, \xi_{sr}(x)) \,\mathrm{d}B_r^k - \int_s^t V_k(r, \xi_{sr}(y)) \,\mathrm{d}B_r^k\right]^p \\
&\leqslant C_p \mathbb{E}\left[\int_s^t |V_k(r, \xi_{sr}(x)) - V_k(r, \xi_{sr}(y))|^2 \,\mathrm{d}r\right]^{p/2} \\
&\leqslant C_p (t-s)^{p/2-1} \mathbb{E}\left[\int_s^t |V_k(r, \xi_{sr}(x)) - V_k(r, \xi_{sr}(y))|^p \,\mathrm{d}r\right] \quad \text{(by Jensen)} \\
&\leqslant C_p M (t-s)^{p/2-1} \mathbb{E}\left[\int_s^t |\xi_{sr}(x) - \xi_{sr}(y)|^p \,\mathrm{d}r\right] \quad \text{(by assumption)} \\
&\leqslant C_p (t-s)^{p/2} |x-y|^p \quad \text{(Lemma 6)}
\end{aligned}$$

and so we can use Kolmogorov again to obtain a version of the stochastic integral which is continuous in x and show that, for fixed $s \leqslant u \leqslant t$, the integral equation

$$\xi_{ut}(x) = x + \sum_{k=0}^{m} \int_u^t V_k(r, \xi_{ur}(x)) \,\mathrm{d}B_r^k$$

is satisfied for all $x \in \mathbb{R}^d$ almost surely. If in this relation we replace x by the random quantity $\xi_{su}(x)$ we obtain

$$\xi_{ut}(\xi_{su}(x)) = \xi_{su}(x) + \sum_{k=0}^{m} \int_u^t V_k(r, \xi_{ur}(\xi_{su}(x))) \,\mathrm{d}B_r^k$$

Let now $\hat{\xi}_{st}(x) = \xi_{ut}(\xi_{su}(x))$ if $t > u$ and $\hat{\xi}_{st}(x) = \xi_{st}(x)$ otherwise. The process $\hat{\xi}_{st}(x)$ satisfies the equation

$$\hat{\xi}_{st}(x) = x + \sum_{k=0}^{m} \int_s^t V_k(r, \hat{\xi}_{sr}(x)) \,\mathrm{d}B_r^k$$

for all $t \geqslant s$ and all $x \in \mathbb{R}^d$ and then by uniqueness of the solution we must have $\xi_{st}(x) = \hat{\xi}_{st}(x)$ a.s. and the statement is proved. □

Theorem 8. *For all $p \geqslant 2$, $0 \leqslant s \leqslant t \leqslant T$, $0 \leqslant s' \leqslant t' \leqslant T, x, x' \in \mathbb{R}^d$:*

$$\mathbb{E}|\xi_{st}(x) - \xi_{s't'}(x')|^p \leqslant C\{|x - x'|^p + (1 + |x| + |x'|)^p(|t - t'|^{p/2} + |s - s'|^{p/2})\}$$

Proof. For simplicity we consider only the case $0 \leqslant s \leqslant s' \leqslant t \leqslant t' \leqslant T$, the others can be obtained via similar reasoning. Using the SDE and Corollary 7 we have

$$\xi_{s't'}(x') = x' + \sum_{k=0}^m \int_{s'}^t V_k(r, \xi_{s'r}(x'))\, \mathrm{d}B_r^k + \sum_{k=0}^m \int_t^{t'} V_k(r, \xi_{s'r}(x'))\, \mathrm{d}B_r^k$$

$$\xi_{st}(x) = \xi_{ss'}(x) + \sum_{k=0}^m \int_{s'}^t V_k(r, \xi_{s'r}(\xi_{ss'}(x)))\, \mathrm{d}B_r^k$$

So

$$\begin{aligned} |\xi_{st}(x) - \xi_{s't'}(x')|^p \leqslant\ & (2m+3)^{p-1}\Bigg\{\underbrace{|\xi_{ss'}(x) - x'|^p}_{A} \\ & + \sum_{k=0}^m \underbrace{\left|\int_{s'}^t [V_k(r, \xi_{s'r}(x')) - V_k(r, \xi_{s'r}(\xi_{ss'}(x)))]\mathrm{d}B_r^k\right|^p}_{B} \\ & + \sum_{k=0}^m \underbrace{\left|\int_t^{t'} V_k(r, \xi_{s'r}(x'))\, \mathrm{d}B_r^k\right|^p}_{C}\Bigg\} \end{aligned}$$

where we used the inequality: $|\sum_{i=1}^N a_i|^p \leqslant N^{p-1} \sum_{i=1}^N |a_i|^p$, which follows from Jensens' inequality. We are going to estimate each of the terms A, B, C separately.

Let us start by an auxiliary result:

$$\begin{aligned} \mathbb{E}|\xi_{ss'}(x) - x|^p &\leqslant (m+3)^{p-1} \sum_{k=0}^m \mathbb{E}\left|\int_s^{s'} V_k(r, \xi_{sr}(x))\, \mathrm{d}B_r^k\right|^p \\ &\leqslant C_p M \mathbb{E}\left(\int_s^{s'} (1 + |\xi_{sr}(x)|)^2\, \mathrm{d}r\right)^{p/2} && \text{(by BDG and the ass. on } V_k) \\ &\leqslant C_p (s' - s)^{p/2}(1 + |x|^p) && \text{(by Jensen and Lemma 6)} \end{aligned}$$

With this estimation we have easily that

$$\mathbb{E}[A] \leqslant 2^{p-1}\{|x - x'|^p + \mathbb{E}|\xi_{ss'}(x) - x|^p\} \leqslant C_p[|x - x'|^p + (s' - s)^{p/2}(1 + |x|^p)]$$

Similar computations leads to

$$\mathbb{E}[B] \leqslant C_p[|x - x'|^p + (s' - s)^{p/2}(1 + |x|^p)], \qquad \text{and} \qquad \mathbb{E}[C] \leqslant C_p(t - t')^{p/2}(1 + |x'|)^p.$$

This is enough to conclude. □

Remark 9. If on the second bound of Lemma 6 we let $\varepsilon \to 0$ we obtain, by monotone convergence,

$$\mathbb{E}|\xi_{st}(x) - \xi_{st}(y)|^{2p} \leqslant C_{p,T}|x - y|^{2p}$$

so if we take $p<0$ we can conclude that if $x\neq y$ then for all t,s we have $\mathbb{P}(\xi_{st}(x)\neq\xi_{st}(y))=1$. We can also show (try) that $\mathbb{P}(\inf_{t\in[s,T]}|\xi_{st}(x)-\xi_{st}(y)|>0)=1$.

Lemma 10. *Let*

$$\eta_{s,t}(x,y)=\frac{1}{|\xi_{s,t}(x)-\xi_{s,t}(y)|}$$

then for any $p>2$ there exists a constant C_p such that

$$\mathbb{E}[|\eta_{s,t}(x,y)-\eta_{s',t'}(x',y')|^p]\leqslant C_p\delta^{-2p}[|x-x'|^p+|y-y'|^p+M(|t-t'|^{p/2}+|s-s'|^{p/2})]$$

with $M=1+|x|^p+|y|^p+|x'|^p+|y'|^p$.

Proof. A simple computation shows that

$$|\eta_{s,t}(x,y)-\eta_{s',t'}(x',y')|^p\leqslant 2^p\eta_{s,t}(x,y)^p\eta_{s',t'}(x',y')^p[|\xi_{s,t}(x)-\xi_{s',t'}(x')|^p+|\xi_{s,t}(y)-\xi_{s',t'}(y')|^p].$$

The reader is invited to complete the proof after taking the expectation of this expression. □

Exercise 3. The previous lemma allows to prove that $x\mapsto\xi_{s,t}(x)$ is injective for all $s<t$ almost surely. This is left as an exercise to the reader.

Lemma 11. *Let $\hat{\mathbb{R}}^d=\mathbb{R}^d\cup\{+\infty\}$ be the one–point compactification of $\mathbb{R}^d$. Let $\hat{x}=x/|x|^2$ and define*

$$\eta_{s,t}(\hat{x})=\begin{cases}\frac{1}{1+|\xi_{s,t}(x)|} & \text{if } \hat{x}\neq 0,\\ 0 & \text{if } \hat{x}=0.\end{cases}$$

Then for any $p>0$ there exists a constant C_p such that

$$\mathbb{E}[|\eta_{s,t}(\hat{x})-\eta_{s,t}(\hat{y})|^p]\leqslant C_p[|\hat{x}-\hat{y}|^p+|t-t'|^{p/2}+|s-s'|^{p/2}].$$

Proof. We note that if x,y are finite

$$|\eta_{s,t}(\hat{x})-\eta_{s,t}(\hat{y})|^p\leqslant\eta_{s,t}(\hat{x})^p\eta_{s,t}(\hat{y})^p|\xi_{s,t}(x)-\xi_{s,t}(y)|^p$$

which can be used to prove the inequality. When $x=\infty$ we have

$$\mathbb{E}[|\eta_{s,t}(\hat{y})|^p]\leqslant C_p(1+|y|)^{-p}\leqslant C_p|\hat{y}|^p$$

so the full inequality follows. □

This lemma allows to prove the onto property for $x\mapsto\xi_{s,t}(x)$. Indeed by Kolmogorov theorem $\hat{x}\mapsto\eta_{s,t}(\hat{x})$ is continuous in a neighborhood of $\hat{x}=0$. Therefore $\xi_{s,t}$ can be extended as a continuous map from $\hat{\mathbb{R}}^d$ to itself for any $s<t$ a.s. Fix one good realization ω, the map $\Xi\colon x\mapsto\xi(\omega)_{s,t}(x)$ is homeomorphic to the identity map $\hat{\mathbb{R}}^d\simeq\mathbb{S}^d\to\hat{\mathbb{R}}^d\simeq\mathbb{S}^d$. By homotopy theory therefore Ξ has to be surjective. Moreover since $\Xi(\infty)=\infty$ also the restriction to $\mathbb{R}^d$ is surjective. The map Ξ^{-1} is continuous bijective since $\hat{\mathbb{R}}^d$ is compact. Then we proven that Ξ is an homeomorphism.

3 Differentiability of the stochastic flow

We say that a function defined over $\mathbb{R}^d$ belongs to $C^{1,\theta}(\mathbb{R}^d)$ if it is differentiable and its derivative is locally Hölder of index θ. We denote $C_g^{1,\theta}(\mathbb{R}^d) \subseteq C^{1,\theta}(\mathbb{R}^d)$ the set of functions whose derivative is globally θ-Hölder.

Theorem 12. *Let us assume that the vectorfields V_k belong to $C_g^{1,\theta}(\mathbb{R}^d)$ uniformly in time and that the derivatives are bounded. Then for all $\theta' < \theta$, $x \mapsto \xi_{st}(x)$ is almost surely of class $C^{1,\theta'}(\mathbb{R}^d)$ uniformly in s, t and the derivative $\nabla \xi_{st}(x)$ satisfies the SDE*

$$\nabla_i \xi_{st}^j(x) = \delta_{ij} + \sum_{k=0}^{m} \sum_{l=1}^{d} \int_s^t \nabla_l V_k^i(r, \xi_{sr}(x)) \nabla_i \xi_{st}^l(x) \mathrm{d}B_r^k \tag{6}$$

for all s, t, x a.s.

Proof. Let

$$\eta_{st}(x, y) = \frac{\xi_{st}(x + y) - \xi_{st}(y)}{|y|}$$

An application of Taylor's formula gives

$$\eta_{st}(x, y) = \frac{y}{|y|} + \int_s^t \underbrace{\int_0^1 \mathrm{d}\tau \nabla V_k(r, \xi_{sr}(x) + \tau |y| \eta_{sr}(x, y))}_{\mathcal{V}_{sr}(x,y)} \eta_{sr}(x, y) \, \mathrm{d}B_r^k .$$

We want to show that $\eta_{st}(x, y)$ is a continuous function of s, t, x, y for all $y \neq 0$ and thus that the limit $y \to 0$ exists and that $\nabla_i \xi_{st}(x) = \lim_{\rho \to 0} \eta_{st}(x, \rho e_i)$ is an Hölder function of x, s, t. All these properties will follow from the following estimation

$$\mathbb{E}|\eta_{st}(x, y) - \eta_{s't'}(x', y')|^p \leqslant C_p\{|x - x'|^{\alpha p} + |y - y'|^{\alpha p} + (1 + |x|^{\alpha p} + |x'|^{\alpha p})[|t - t'|^{\alpha p/2} + |s - s'|^{\alpha p/2}]\}. \tag{7}$$

Let us prove eq. (7) step by step. First step, boundedness of $\eta_{st}(x, y)$ in L^p:

$$\mathbb{E}|\eta_{st}(x, y)|^p \leqslant C_p \left[1 + |t - s|^{p/2-1} \sup_{z,u,k} |\nabla V_k(u, z)|^p \int_s^t \mathbb{E}|\eta_{sr}(x, y)|^p \mathrm{d}u \right] \tag{8}$$

by Gronwall we have $\mathbb{E}|\eta_{st}(x, y)|^p \leqslant C_p \, e^{C_p T^{p/2}}$ for all $0 \leqslant s \leqslant t \leqslant T$. Consider now the case $t = t'$ and $s \leqslant s' \leqslant t'$

$$\eta_{st}(x, y) - \eta_{s't'}(x', y') = \underbrace{\int_s^{s'} \mathcal{V}_{sr}(x, y) \eta_{sr}(x, y) \, \mathrm{d}B_r}_{A} + \underbrace{\int_{s'}^t [\mathcal{V}_{sr}(x, y) \eta_{sr}(x, y) - \mathcal{V}_{s'r}(x', y') \eta_{s'r}(x', y')] \mathrm{d}B_r}_{B}.$$

Then

$$\mathbb{E}[|\mathcal{A}|^p] \leqslant C_p |s - s'|^{p/2-1} \int_s^{s'} \mathbb{E}|\eta_{sr}(x, y)|^p \, \mathrm{d}r \leqslant C_p |s - s'|^{p/2}.$$

The integrand of B is estimated by

$$|\mathcal{V}_{sr}(x,y)-\mathcal{V}_{s'r}(x',y')|\,|\eta_{sr}(x,y)|+|\mathcal{V}_{s'r}(x',y')|\,|\eta_{sr}(x,y)-\eta_{s'r}(x',y')|$$

$$\leqslant C\|\nabla V\|_{\infty}|\eta_{sr}(x,y)-\eta_{s'r}(x',y')|$$

$$+C(|\,\xi_{sr}(x)-\xi_{s'r}(x')|^{\alpha}+|\,\xi_{sr}(x+y)-\xi_{s'r}(x'+y')|^{\alpha})|\,\eta_{s'r}(x',y')|.$$

We can then control $\mathbb{E}[|B|^p]$ by

$$C[(1+|x|^{\alpha p}+|x'|^{\alpha p})|s-s'|^{\alpha p/2}+|x-x'|^{\alpha p}+|y-y'|^{\alpha p}]+C\int_{s'}^{t}\mathbb{E}|\eta_{sr}(x,y)-\eta_{s'r}(x',y')|^p\mathrm{d}r$$

and applying Gronwall we deduce that

$$\mathbb{E}|\eta_{st}(x,y)-\eta_{s't'}(x',y')|^p\leqslant\mathcal{D}+C\int_{s'}^{t}\mathbb{E}|\eta_{sr}(x,y)-\eta_{s'r}(x',y')|^p\mathrm{d}r$$

where $\mathcal{D}:=C'[(1+|x|^{\alpha p}+|x'|^{\alpha p})|s-s'|^{\alpha p/2}+|x-x'|^{\alpha p}+|y-y'|^{\alpha p}]$. This gives us eq. (7). We need now to control the case $t<t'$ (all the others can be dealt with similarly). We have

$$\eta_{st}(x,y)-\eta_{s't'}(x',y')=\eta_{st}(x,y)-\eta_{s't}(x',y')+\int_{t}^{t'}\mathcal{V}_{s'r}(x',y')\,\eta_{s'r}(x',y')\,\mathrm{d}B_r$$

and thanks for the bound (8) the stochastic integral can be bounded in L^p by $C|t-t'|^{1/2}$ and we conclude easily. □

Remark 13. If we assume that $V_k\in C_g^{n,\theta}$ then, by similar methods, we can obtain $\xi_{st}\in C^{n,\theta'}$ for all $0<\theta\text{'}<\theta$.

Theorem 14. *Almost surely, the Jacobian matrix $\nabla\xi_{st}(x)$ is not singular for all s,t,x.*

Proof. By Theorem 12 the matrix $\nabla\xi$ satisfies the integral equation (6). We consider then the following equation for the process $t\mapsto K_{st}(x)$ with values in $d\times d$ matrices:

$$d_t\,K_{st}(x)=-K_{st}(x)\,\nabla V_k(\xi_{st}(x))\,\mathrm{d}B_t^k-K_{st}(x)\nabla V_k\,(\xi_{st}(x))\,\nabla V_k(\xi_{st}(x))\,\mathrm{d}t,\qquad t>s,$$

with initial condition $K_{ss}(x)=I_{d\times d}$. It is easy to see that this equation has a unique solution which is global in time and continuous in s,t,x if $V\in C_g^{1,\alpha}$. Itô formula gives

$$d_t[K_{st}(x)\,\nabla\xi_{st}(x)]=[d_tK_{st}(x)]\nabla\xi_{st}(x)+K_{st}(x)\,[d_t\nabla\xi_{st}(x)]+d_t\langle K_{s\cdot}(x),\nabla\xi_{s\cdot}(x)\rangle=0$$

and then $K_{st}(x)\nabla\xi_{st}(x)=K_{ss}(x)\nabla\xi_{ss}(x)=I_{d\times d}$ for all $t\geqslant s$ and x. This shows that the matrix $\nabla\xi_{st}(x)$ is not singular and that $[\nabla\xi_{st}(x)]^{-1}=K_{st}(x)$. □

4 Backward Stochastic integrals

We already seen that the flow ξ driven by B and V_k satisfies the Itô formula

$$F(\xi_{st}(x))=F(x)+\sum_{k=0}^{m}\int_{s}^{t}\mathcal{V}_k(r)F(\xi_{sr}(x))\,\mathrm{d}B_r^k+\int_{s}^{t}\mathcal{L}(r)F(\xi_{sr}(x))\mathrm{d}r$$

where

$$\mathcal{V}_k(r)F(x) = \sum_{i=1}^{d} V_k^i(r,x)\frac{\partial F(x)}{\partial x_i}, \qquad \mathcal{L}(r)F(x) = \sum_{i,j=1}^{d}\sum_{k=1}^{m} V_k^i(r,x)V_k^j(r,x)\frac{\partial^2 F(x)}{\partial x_i \partial x_j}.$$

Note that the coefficients of the vectorfields are computed along the flow and that the variable s is kept fixed while the variable t is subject of the stochastic calculus.

We are going now to obtain another representation of $F(\xi_{st}(x))$ as a *backward* semimartingale in the variable s, keeping t fixed.

Theorem 15. *If $V_k \in C^{2,\alpha}(\mathbb{R}^d;\mathbb{R}^d)$ and $F \in C^2(\mathbb{R}^d;\mathbb{R})$ then*

$$F(\xi_{st}(x)) - F(x) = \sum_{k=0}^{m} \int_s^t \mathcal{V}_k(r)(F\circ\xi_{rt})(x)\hat{\mathrm{d}}B_r^k + \int_s^t \mathcal{L}(r)(F\circ\xi_{rt})(x)\mathrm{d}r.$$

Proof. We fix a partition $\Delta = \{0 = s_0 < s_1 < \cdots < s_n = t\}$ of $[0,t]$ and we assume that $s = s_\ell$ for some $0 \leqslant \ell \leqslant n$. Then

$$F(\xi_{st}(x)) - F(x) = \sum_{k=\ell}^{n-1} [(F\circ\xi_{s_{k+1}t})(\xi_{s_k s_{k+1}}(x)) - (F\circ\xi_{s_{k+1}t})(x)]. \tag{9}$$

A Taylor expansion gives the this quantity is equal to

$$\sum_{k=\ell}^{n-1}\sum_{i=1}^{d} \nabla_i(F\circ\xi_{s_{k+1}t})(x)\,[\xi^i_{s_k s_{k+1}}(x) - x_i]$$

$$+\sum_{k=\ell}^{n-1}\sum_{i,j=1}^{d} \nabla_{ij}^2(F\circ\xi_{s_{k+1}t})(x+\eta_k)\,[\xi^i_{s_k s_{k+1}}(x) - x_i][\xi^j_{s_k s_{k+1}}(x) - x_j]$$

where η_k are r.v. on $\mathbb{R}^d$ such that $|\eta_k| \leqslant |\,\xi_{s_k s_{k+1}}(x) - x|$. We will show the following convergences

$$\mathcal{A}^\Delta = \sum_{k=\ell}^{n-1}\sum_{i=1}^{d} \nabla_i(F\circ\xi_{s_{k+1}t})(x)\,[\xi^i_{s_k s_{k+1}}(x) - x_i] \to \sum_{j=0}^{m}\int_s^t \mathcal{V}_j(r)(F\circ\xi_{rt})(x)\hat{\mathrm{d}}B_r^j$$

$$\mathcal{B}^\Delta = \sum_{k=\ell}^{n-1}\sum_{i,j=1}^{d} \nabla_{ij}^2(F\circ\xi_{s_{k+1}t})(x)\,[\xi^i_{s_k s_{k+1}}(x) - x_i][\xi^j_{s_k s_{k+1}}(x) - x_j] \to \int_s^t \mathcal{L}(r)(F\circ\xi_{rt})(x)\mathrm{d}r$$

and moreover

$$\mathcal{C}^\Delta = \sum_{k=\ell}^{n-1}\sum_{i,j=1}^{d} [\nabla_{ij}^2(F\circ\xi_{s_{k+1}t})(x+\eta_k) - \nabla_{ij}^2(F\circ\xi_{s_{k+1}t})(x)]\,[\xi^i_{s_k s_{k+1}}(x) - x_i][\xi^j_{s_k s_{k+1}}(x) - x_j] \to 0$$

when the size $|\Delta|$ of the partition Δ goes to zero. This is enough to prove eq. (9).

We have

$$\mathcal{A}^\Delta = \sum_{j=0}^{m}\sum_{k=\ell}^{n-1}\sum_{i=1}^{d} \nabla_i(F\circ\xi_{s_{k+1}t})(x)\int_{s_k}^{s_{k+1}} V_j^i(r,\xi_{s_k r}(x))\,\mathrm{d}B_r^j = \sum_{j=0}^{m}\sum_{i=1}^{d}\mathcal{A}_{ij}^\Delta$$

For $r \in [0, t]$ and $j \geqslant 1$ let I_r^{Δ} the process

$$I_r^{\Delta} := \mathbb{E}[\mathcal{A}_{ij}^{\Delta} | \mathcal{F}_{rt}]$$

which is a continuous and square integrable backward martingale. Let now M_r^{Δ} the continuous and square integrable backward martingale defined by

$$M_r^{\Delta} := \mathbb{E}[\tilde{\mathcal{A}}_{ij}^{\Delta} | \mathcal{F}_{rt}]$$

where

$$\tilde{\mathcal{A}}_{ij}^{\Delta} = \sum_{k=0}^{n-1} \nabla_i (F \circ \xi_{s_{k+1} t})(x)\, V_j^i(r, x)(B_{s_{k+1}}^j - B_{s_k}^j)$$

A direct computation shows that the quadratic variation of $I^{\Delta} - M^{\Delta}$ along the partition Δ is given by

$$\langle I^{\Delta} - M^{\Delta} \rangle_t^{\Delta} = \sum_{k=0}^{n-1} |\nabla_i (F \circ \xi_{s_{k+1} t})(x)| \left(\int_{s_k}^{s_{k+1}} [V_j^i(r, \xi_{s_k r}(x)) - V_j^i(r, x)] \,\mathrm{d} B_r^j \right)^2$$

which is going to zero in L^1 when $|\Delta| \to 0$. So $I_r^{\Delta} - M_r^{\Delta}$ converges to zero uniformly in L^2. But now $M_r^{\Delta} \to M_r$ where

$$M_r := \int_r^t \nabla_i (F \circ \xi_{rt})(x)\, V_j^i(r, x) \hat{\mathrm{d}}\, B_r^j.$$

It is easy to show the convergence of the terms with $j = 0$.

Let

$$J_r^{\Delta, v} = \mathbb{E}\left[\sum_{k=\ell}^{n-1} \nabla_{ij}^2 (F \circ \xi_{s_{k+1} t})(x) \int_{s_k}^{s_{k+1}} V_v^i(r, \xi_{s_k r}(x)) \,\mathrm{d} B_r^v \middle| \mathcal{F}_{rt} \right]$$

and

$$K_r^{\Delta, v} = \mathbb{E}\left[\sum_{k=\ell}^{n-1} \int_{s_k}^{s_{k+1}} V_v^j(r, \xi_{s_k r}(x)) \,\mathrm{d} B_r^v \middle| \mathcal{F}_{rt} \right]$$

then $\mathcal{B}^{\Delta} = \sum_{v=1}^m [\langle J^{\Delta, v}, K^{\Delta, v} \rangle_t^{\Delta} - \langle J^{\Delta, v}, K^{\Delta, v} \rangle_s^{\Delta}]$. Again, it is easy to show that $J^{\Delta, v}$ and $K^{\Delta, v}$ converge to J_r^v and K_r^v respectively, where

$$J_r^v := \int_r^t \nabla_{ij}^2 (F \circ \xi_{rt})(x)\, V_v^i(r, x) \hat{\mathrm{d}}\, B_r^v$$

$$K_r^v := \int_r^t V_v^j(r, x) \hat{\mathrm{d}}\, B_r^v.$$

We are going to show that $\langle J^{\Delta, v}, K^{\Delta, v} \rangle_t^{\Delta} \to \langle J^v, K^v \rangle_t$ giving the convergence we are looking after. We have

$$|\langle J^{\Delta, v}, K^{\Delta, v} \rangle_t^{\Delta} - \langle J^v, K^v \rangle_t| \leqslant |\langle J^{\Delta, v}, K^{\Delta, v} \rangle_t^{\Delta} - \langle J^v, K^v \rangle_t^{\Delta}| + |\langle J^v, K^v \rangle_t^{\Delta} - \langle J^v, K^v \rangle_t|$$

and it is clear that the second term is going to 0. For the first we use that

$$|\langle J^{\Delta, v}, K^{\Delta, v} \rangle_t^{\Delta} - \langle J^v, K^v \rangle_t^{\Delta}| \leqslant (\langle J^{\Delta, v} - J^v \rangle_t^{\Delta} \langle K^{\Delta, v} \rangle_t^{\Delta})^{1/2} + (\langle K^{\Delta, v} - K^v \rangle_t^{\Delta} \langle J^{\Delta, v} \rangle_t^{\Delta})^{1/2}.$$

If we now observe that $|J_t^{\Delta,v} - J_t^v|^2 - \langle J^{\Delta,v} - J^v\rangle_t^\Delta$ is a continuous martingale, we have

$$\mathbb{E}\sup_t \langle J^{\Delta,v} - J^v\rangle_t^\Delta \leqslant 17\mathbb{E}|J_t^{\Delta,v} - J_t^v|^2 \to 0.$$

Moreover

$$|\mathcal{C}^\Delta| \leqslant \sup_k |\nabla_{ij}^2(F\circ\xi_{s_{k+1}t})(x+\eta_k) - \nabla_{ij}^2(F\circ\xi_{s_{k+1}t})(x)\{\langle K^\Delta(i)\rangle_t^\Delta \langle K^\Delta(j)\rangle_t^\Delta\}^{1/2} \to 0$$

and we are done. □

Bibliography

[1] H. Kunita. Stochastic differential equations and stochastic flows of diffeomorphisms. In P. L. Hennequin, editor, *École d'Été de Probabilités de Saint-Flour XII - 1982*, number 1097 in Lecture Notes in Mathematics, pages 143–303. Springer Berlin Heidelberg, 1984. DOI: 10.1007/BFb0099433.

[2] Hiroshi Kunita. *Stochastic Flows and Stochastic Differential Equations*. Cambridge University Press, Cambridge; New York, apr 1997.

[3] Stanisław Kwapień and Jan Rosiński. Sample Hölder Continuity of Stochastic Process and Majorizing Measures. In Robert C. Dalang, Marco Dozzi, and Francesco Russo, editors, *Seminar on Stochastic Analysis, Random Fields and Applications IV*, number 58 in Progress in Probability, pages 155–163. Birkhäuser Basel, 2004. DOI: 10.1007/978-3-0348-7943-9_11.

[4] Daniel W. Stroock. *Probability Theory: An Analytic View*. Cambridge University Press, Cambridge ; New York, 2 edition edition, dec 2010.