Stochastic Analysis – Problem Sheet 1.

Tutorial classes: Mon 25th April 12–14, Wed 27th April 16–18 in SemR 0.006. Philipp Boos <s6phboos@uni-bonn.de>.
Solutions will be collected Thursday 21st April during the lecture. At most in groups of 3.

Exercise 1. (Martingale problem) Consider the solution X of the SDE in $\mathbb{R}^d$

$$\mathrm{d}X_t = b(X_t)\mathrm{d}t + \sigma(X_t)\mathrm{d}B_t,$$

where B is a d-dimensional Brownian motion and $b:\mathbb{R}^d \to \mathbb{R}^d$, $\sigma:\mathbb{R}^d \to \mathbb{R}^{d\times d}$ locally bounded coefficients. Let $\mathcal{L}$ be the associated infinitesimal generator. By Theorem 2.3 in [Eberle, Stochastic Analysis notes SS2015] we know that the following two conditions are equivalent

i. For any $f \in C^2(\mathbb{R}^d)$, the process $M_t^f = f(X_t) - f(X_0) - \int_0^t \mathcal{L}f(X_s)\mathrm{d}s$ is a local martingale.

ii. For any $v \in \mathbb{R}^d$, the process $M_t^v = v \cdot X_t - v \cdot X_0 - \int_0^t v \cdot b(X_s)\mathrm{d}s$ is a local martingale with quadratic variation

$$[M^v]_t = \int_0^t v \cdot a(X_s) v \,\mathrm{d}s.$$

a) Show that these conditions are also equivalent to the fact that for any $v \in \mathbb{R}^d$ the process

$$Z_t^v = \exp\left(M_t^v - \frac{1}{2}\int_0^t v \cdot a(X_s) v \,\mathrm{d}s \right)$$

is a local martingale. *[Hint: use the fact that linear combinations of exponentials are dense in C^2 w.r.t. uniform convergence on compacts for the functions and its first two derivatives (assumed wihtout proof)]*

b) Show that these conditions imply that

$$(f(X_t)/f(X_0))\exp\left(-\int_0^t \frac{\mathcal{L}f}{f}(X_s)\mathrm{d}s \right)$$

is a local martingale for every stricly positive C^2 function f.

Exercise 2. (Variation of constants) Consider the nonlinear SDE

$$\mathrm{d}X_t = f(t, X_t)\mathrm{d}t + c(t) X_t \mathrm{d}B_t, \qquad X_0 = x,$$

where $f:\mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$ and $c:\mathbb{R}_+ \to \mathbb{R}$ are continuous deterministic functions.

a) Find an explicit solution Z_t in the case $f = 0$ and $Z_0 = 1$.

b) Use the Ansatz $X_t = C_t Z_t$ to show that X solves the SDE provided C solves an ODE with random coefficients.

c) Apply this method to solve the SDE

$$\mathrm{d}X_t = X_t^{-1}\mathrm{d}t + \alpha X_t \mathrm{d}B_t, \qquad X_0 = x$$

where α is a constant.

d) Apply the method to study the solution of the SDE

$$\mathrm{d}X_t = X_t^{\gamma}\mathrm{d}t + \alpha X_t \mathrm{d}B_t, \qquad X_0 = x > 0$$

where α and γ are constants. For which values of γ do we get explosion?

Exercise 3. (Exit distribution of Bessel process) Let X be the solution of the SDE

$$\mathrm{d}X_t = \frac{d-1}{2}\frac{1}{X_t}\mathrm{d}t + \mathrm{d}B_t \qquad X_0 = x_0 > 0$$

where B is a standard Brownian motion and $d > 1$ is a constant.

a) Find a non–constant function u such that $u(X_t)$ is a local martingale.

b) Compute the ruin probability $\mathbb{P}(T_a < T_b)$ for $0 < a < b$ with $x_0 \in [a, b]$ where $T_a = \inf\{t \geqslant 0 \colon X_t \leqslant a\}$ and $T_b = \inf\{t \geqslant 0 \colon X_t \geqslant b\}$.

c) Proceed similarly to determine the mean exit time $\mathbb{E}[T]$ where $T = \min(T_a, T_b)$.