Stochastic Analysis – Problem Sheet 3.

Tutorial classes: Mon May 9th 16–18 in SemR 0.008. Philipp Boos <`s6phboos@uni-bonn.de`>.
Solutions will be collected at the beginning of the tutorial session. At most in groups of 3.

Exercise 1. (Passage time to a sloping line) Let X be a one–dimensional Brownian motion with $X_0 = 0$ and let $a > 0$, $b \in \mathbb{R}$.

a) Let $T_L = \inf \{t \geqslant 0: X_t = a + b\,t\}$ denote the first passage time to the line $y = a + b\,t$. Show that

$$\mathbb{P}(T_L \leqslant t) = \mathbb{E}[e^{-bX_t - b^2 t/2} \mathbb{I}_{T_a \leqslant t}], \tag{1}$$

where $T_a = \inf \{t \geqslant 0: X_t = a\}$ is the first passage time to level a.

b) Recall that, by the reflection principle, the law of T_a is absolutely continuous with density

$$f_{T_a}(t) = a\,t^{-3/2} \varphi(a / \sqrt{t}) \mathbb{I}_{(0,\infty)}(t),$$

where φ is the standard normal density. Deduce that the law of T_L is absolutely continuous with density

$$f_{T_L}(t) = a\,t^{-3/2} \varphi((a + b\,t) / \sqrt{t}) \mathbb{I}_{(0,\infty)}(t).$$

[Hint: in (1) take the conditional expectation w.r.t. $\mathcal{F}_{T_a}$].

c) Show that, for $b > 0$,

$$\mathbb{E}\Big[e^{-bX_t} \max_{s \leqslant t} (X_s)\Big] \simeq \frac{e^{b^2 t/2}}{2b}, \quad \text{and} \quad \mathbb{E}\Big[e^{bX_t} \max_{s \leqslant t} (X_s)\Big] \simeq b\,\mathrm{t}e^{b^2 t/2}, \qquad \text{as } t \to \infty.$$

Exercise 2. (Brownian motion writes your name) Prove that a Brownian motion in $\mathbb{R}^2$ will write your name (in cursive script, without dotted 'i's or crossed 't's). Let B be a two dimensional Brownian motion on $[0,1]$ and observe that $X_t^{(a,b)} = (b-a)^{1/2}(B_{a+(b-a)t} - B_a)$ for $t \in [0,1]$ has the same law as B. Let $g: [0,1] \to \mathbb{R}^2$ a parametrization of your name and note that the Brownian motion $X^{(a,b)}$ spells your name (to precision $\varepsilon > 0$) if

$$\sup_{t \in (0,1)} \big|X_t^{(a,b)} - g(t)\big| \leqslant \varepsilon. \tag{2}$$

a) For $k \in \mathbb{N}$ let A_k be the event that (2) holds for $a = 2^{-k-1}$ and $b = 2^{-k}$. Check that the events $(A_k)_{k \in \mathbb{N}}$ are independent and $\mathbb{P}(A_k) = \mathbb{P}(A_0)$ for all $k \geqslant 0$. Conclude that if $\mathbb{P}(A_0) > 0$ then infinitely many of the A_ks will occur almost surely.

b) Show that

$$\mathbb{P}\Big[\sup_{t \in (0,1)} |B_t| \leqslant \varepsilon\Big] > 0. \tag{3}$$

c) Using (3) and Girsanov's transform to show that $\mathbb{P}(A_0) > 0$.

Exercise 3. (Brownian Bridge) Let X be a d–dimensional Brownian motion with $X_0 = 0$.

a) Show that, for any $y \in \mathbb{R}^d$, the process

$$X_t^y = X_t - t\,(X_1 - y) \qquad t \in [0, 1]$$

is independent of X_1.

b) Let μ_y denote the law of X^y on $C([0, 1]; \mathbb{R}^d)$. Show that $y \mapsto \mu_y$ is a regular version of the conditional distribution of X given $X_1 = y$.