Institute for Applied Mathematics – SS2016
Massimiliano Gubinelli / Philipp Boos

Stochastic Analysis – Problem Sheet 5.

Tutorial classes: Mon May 30th in SemR 0.008. Philipp Boos <`s6phboos@uni-bonn.de`>.
Solutions will be collected at the beginning of the tutorial session. At most in groups of 3.

Exercise 1. Let M be a positive continuous supermartingale such that $\mathbb{E}[M_0] < \infty$. Let $M_\infty = \lim_{t\to\infty} M_t$ (assumed to exist $\mathbb{P}$–a.s.). Show that if $\mathbb{E}[M_\infty] = \mathbb{E}[M_0]$ then M is a martingale and $\mathbb{E}[M_\infty|\mathcal{F}_t] = M_t$. *[Hint: prove that $\mathbb{E}[M_\infty|\mathcal{F}_t] \leqslant M_t$ and that $\mathbb{E}[M_t] = \mathbb{E}[M_0]$ and conclude.]*

Exercise 2. Assume that $\Omega = C(R_{\geqslant 0}; \mathbb{R}^d)$, $\mathbb{P}$ is the d–dimensional Wiener measure and that X is the canonical process on Ω and that the filtration $\mathcal{F}_\bullet$ is generated by X. Consider a predictable $\mathbb{R}^d$-valued drift b given by a function $b: \mathbb{R}_{\geqslant 0} \times \Omega \to \mathbb{R}^d$. By tilting $\mathbb{P}$ via $Z = \mathcal{E}(\int_0^\cdot b(X)\mathrm{d}X)$ we obtain that, under the tilted measure $\mathbb{P}^b$ the process X is a solution of the SDE

$$\mathrm{d}X_t = b_t(X) + \mathrm{d}W_t, \qquad t \geqslant 0$$

where W is a $\mathbb{P}^b$–Brownian motion.

a) Prove that if

$$|b_t(x)| \leqslant C(1 + |x_t|), \qquad t \geqslant 0, x \in \Omega,$$

then Novikov's condition holds conditionally on $\mathcal{F}_s$ for intervals $[s, t]$ such that $|t - s|$ is small enough, i.e.

$$\mathbb{E}\left[\exp\left(\frac{1}{2}\int_s^t |b_u(X)|^2 \mathrm{d}u\right)|\mathcal{F}_s\right] < +\infty.$$

b) Deduce that Z is a martingale. [Hint: prove that $\mathbb{E}[Z_t|\mathcal{F}_s] = Z_s$ for small time intervals $[s, t]$ and the conclude].

c) Prove that

$$\mathbb{P}(\|X\|_{[0,t]} > r) \leqslant 2\, d e^{-r^2/2dt} \qquad t \geqslant 0, r \geqslant 0.$$

where $\|X\|_{[0,t]}$ denotes the supremum wrt. the Euclidean norm of $(X_s)_{s\in[0,t]}$.

[Hint: use Doob's inequality for the submartingale $e^{\lambda X_t^i}$ and optimize over $\lambda > 0$]

d) Prove the same result as in (a) under the more general assumption that b is a previsible drift such that

$$|b_t(x)| \leqslant C(1 + \|x\|_{\infty,[0,t]}), \qquad t \geqslant 0, x \in \Omega$$

where $C < +\infty$.