Institute for Applied Mathematics – SS2016
Massimiliano Gubinelli / Philipp Boos

Stochastic Analysis – Problem Sheet 9.

Tutorial classes: Mon July 11th in SemR 0.008. Philipp Boos <`s6phboos@uni-bonn.de`>.
Solutions will be collected at the beginning of the tutorial session. At most in groups of 3.

Exercise 1. Let $(B_t)_{t\in[0,1]}$ be a standard Brownian motion. Note that if $\lambda < 1/2$ we have

$$\mathbb{E}\int_{[0,1]^2} \mathrm{d}t\,\mathrm{d}s\, e^{\lambda\frac{|B_t-B_s|^2}{|t-s|}} < +\infty.$$

Use this observation and the Garsia–Rodemich–Rumsey inequality with $\Psi(x) = e^{\lambda x^2} - 1$ to derive an almost sure modulus of continuity for Brownian motion.

Exercise 2. Prove (the upper bound of) Burkholder–Davis–Gundy inequality. Let M be a continuous local martingale (with $M_0 = 0$). For any $p \geqslant 2$ we have

$$\mathbb{E}[\sup_{t\leqslant T} |M_t|^p] \leqslant C_p \mathbb{E}\big[\big([M]_T^{p/2}\big)\big]$$

where C_p is a universal constant depending only on p.

a) Assume that the martingale M is bounded. Use Itô formula on $t \mapsto (\varepsilon + |M_t|^2)^{p/2}$ to prove that

$$\mathbb{E}[\sup_{t\leqslant T} |M_t|^p] \leqslant \int_0^T \mathbb{E}[|M_t|^{p-2}\mathrm{d}[M]_t].$$

(why we need $\varepsilon > 0$?)

b) Use Hölder's and Doob's inequality to conclude.

c) Remove the assumption of boundedness.

Exercise 3. Let us continue with the setting of Exercise 1 and prove now a complementary lower bound when $p \geqslant 4$, that is

$$\mathbb{E}\big[\big([M]_T^{p/2}\big)\big] \leqslant C_p \mathbb{E}[\sup_{t\leqslant T} |M_t|^p].$$

where again C_p is a universal constant depending only on p (not the same as that of the upper bound).

a) Use the relation

$$[M]_T = M_T^2 - 2\int_0^T M_s \mathrm{d}M_s$$

to estimate $\mathbb{E}\big[\big([M]_T^{p/2}\big)\big]$ and then use the BDG upper bound for the stochastic integral.

b) Prove that if we let $N_T = \int_0^T M_s \mathrm{d}M_s$ then for any $\varepsilon > 0$ there exists $\lambda_\varepsilon > 0$ such that

$$[N]_T^{1/2} \leqslant \lambda_\varepsilon \sup_{t\leqslant T} |M_t| + \varepsilon [M]_T$$

c) Conclude by choosing ε small enough.

Exercise 4. Prove a substitution lemma for stochastic integrals. Let us be given a standard basis $(\Omega, \mathcal{F}, \mathcal{F}_\bullet, \mathbb{P})$ and let Y be an $\mathcal{F}_0$ measurable r.v. with values in $\mathbb{R}^d$ and $f: \Omega \times \mathbb{R}_+ \times \mathbb{R}^d \to \mathbb{R}$ a stochastic process such that for any $x \in \mathbb{R}^d$ the process $s \mapsto f_s(x)$ is predictable, $\int_0^T |f_s(x)|^2 \mathrm{d}[M]_s < \infty$ a.s. and $(s, x) \mapsto f_s(x)$ is continuous. Let M be a continuous local martingale. Then

$$\left[\int_0^T f_s(x) \mathrm{d}M_s\right]\bigg|_{x=Y} = \int_0^T f_s(Y) \mathrm{d}M_s.$$

a) Consider a partition Δ of $[0, T]$ and let $f_\cdot^\Delta(x)$ be piecewise linear approximations to $f_\cdot(x)$. Prove that for any compact $K \subset \mathbb{R}^d$, $\int_0^T \sup_{x \in K} |f_s(x) - f_s^\Delta(x)|^2 \mathrm{d}[M]_s \to 0$ a.s. as the size of the partition $|\Delta| \to 0$.

b) Deduce that there exists a sequence of partitions $(\Delta_n)_{n \geqslant 1}$ such that $|\Delta_n| \to 0$ and that, letting $J^n(x) := \int_0^T f_s^{\Delta_n}(x) \mathrm{d}M_s$ and $J(x) := \int_0^T f_s(x) \mathrm{d}M_s$ we have for any compact K

$$\sup_{x \in K} |J^n(x) - J(x)| \to 0 \qquad \text{a.s.}$$

c) Prove that the substitution formula is true for f^Δ and conclude.

Exercise 5. (Bonus) Prove a Fubini theorem for stochastic integrals. Let $(\Lambda, \mathcal{A})$ be a measure space and $(\Omega, \mathcal{F}, \mathcal{F}_\bullet, \mathbb{P})$ a filtered probability space.

a) Let $(X_n)_n$ a sequence of functions $X_n: \Omega \times \Lambda \to \mathbb{R}$ which are $\mathcal{F} \otimes \mathcal{A}$ measurable (product σ-field) and such that $(X_n(\cdot, \lambda))_n$ converges in probability for any fixed $\lambda \in \Lambda$. Prove that there exists an $\mathcal{F} \otimes \mathcal{A}$ measurable function $X: \Omega \times \Lambda \to \mathbb{R}$ for which $X_n(\cdot, \lambda) \xrightarrow[\mathbb{P}]{} X(\cdot, \lambda)$ for any $\lambda \in \Lambda$. [Hint: here the difficulty is the measurability of the limit X, consider the sequence $n_k(\lambda)$ defined by $n_0(\lambda) = 1$ and

$$n_{k+1}(\lambda) = \inf\left\{m > n_k(\lambda): \sup_{p, q \geqslant m} \mathbb{P}[|X_p(\cdot, \lambda) - X_q(\cdot, \lambda)| > 2^{-k}] \leqslant 2^{-k}\right\}$$

and prove that $\lim_k X_{n_k(\lambda)}(\cdot, \lambda)$ exists a.s. and conclude]

b) Let $H: \Lambda \times \mathbb{R}_{\geqslant 0} \times \Omega \to R$ be a bounded function which is measurable w.r.t. $\mathcal{A} \otimes \mathcal{P}$ where $\mathcal{P}$ is the predictable σ-field on $\mathbb{R}_{\geqslant 0} \times \Omega$. Let M be a continuous martingale on $(\Omega, \mathcal{F}, \mathcal{F}_\bullet, \mathbb{P})$. Prove that there exists a function $J: \Lambda \times \Omega \to \mathbb{R}$ measurable for $\mathcal{A} \otimes \mathcal{F}_T$ which is a version of the stochastic process $\lambda \mapsto J(\lambda) := \int_0^T H(\lambda, s) \mathrm{d}M_s$ and for which it holds

$$\int_\Lambda J(\lambda) m(\mathrm{d}\lambda) = \int_0^T \left[\int_\Lambda H(\lambda, s, \cdot)\, m(\mathrm{d}\lambda)\right] \mathrm{d}M_s, \qquad a.s.$$

for any bounded measure m on $(\Lambda, \mathcal{A})$. Hint: prove that

$$\mathbb{E}\left[\left(\int_0^T \left[\int_\Lambda H(\lambda, s, \cdot)\, m(\mathrm{d}\lambda)\right] \mathrm{d}M_s - \int_\Lambda J(\lambda) m(\mathrm{d}\lambda)\right)^2\right] = 0.$$