

Stochastic Analysis – Problem Sheet 1.

Tutorial classes: Mon May 8th 16–18 in SemR 1.007. Claudio Bellani <claudio.bellani01@gmail.com>. Solutions will be collected Thursday May 2nd during the lecture. At most in groups of 3.

Exercise 1. (Martingale problem) Consider the solution X of the SDE in \mathbb{R}^d

$$dX_t = b(X_t)dt + \sigma(X_t)dB_t,$$

where B is a d -dimensional Brownian motion and $b: \mathbb{R}^d \rightarrow \mathbb{R}^d, \sigma: \mathbb{R}^d \rightarrow \mathbb{R}^{d \times d}$ locally bounded coefficients. Let \mathcal{L} be the associated infinitesimal generator. By Theorem 2.3 in [Eberle, Stochastic Analysis notes SS2015] we know that the following two conditions are equivalent

- i. For any $f \in C^2(\mathbb{R}^d)$, the process $M_t^f = f(X_t) - f(X_0) - \int_0^t \mathcal{L}f(X_s)ds$ is a local martingale.
- ii. For any $v \in \mathbb{R}^d$, the process $M_t^v = v \cdot X_t - v \cdot X_0 - \int_0^t v \cdot b(X_s)ds$ is a local martingale with quadratic variation

$$[M^v]_t = \int_0^t v \cdot a(X_s)v ds.$$

- a) Show that these conditions are also equivalent to the fact that for any $v \in \mathbb{R}^d$ the process

$$Z_t^v = \exp\left(M_t^v - \frac{1}{2} \int_0^t v \cdot a(X_s)v ds\right)$$

is a local martingale. [Hint: use the fact that linear combinations of exponentials are dense in C^2 w.r.t. uniform convergence on compacts for the functions and its first two derivatives (assumed without proof)]

- b) Show that these conditions imply that

$$(f(X_t) / f(X_0)) \exp\left(- \int_0^t \frac{\mathcal{L}f}{f}(X_s)ds\right)$$

is a local martingale for every strictly positive C^2 function f .

Exercise 2. (Variation of constants) Consider the nonlinear SDE

$$dX_t = f(t, X_t)dt + c(t)X_t dB_t, \quad X_0 = x,$$

where $f: \mathbb{R}_+ \times \mathbb{R} \rightarrow \mathbb{R}$ and $c: \mathbb{R}_+ \rightarrow \mathbb{R}$ are continuous deterministic functions.

- a) Find an explicit solution Z_t in the case $f = 0$ and $Z_0 = 1$.
- b) Use the Ansatz $X_t = C_t Z_t$ to show that X solves the SDE provided C solves an ODE with random coefficients.

c) Apply this method to solve the SDE

$$dX_t = X_t^{-1}dt + \alpha X_t dB_t, \quad X_0 = x$$

where α is a constant.

d) Apply the method to study the solution of the SDE

$$dX_t = X_t^\gamma dt + \alpha X_t dB_t, \quad X_0 = x > 0$$

where α and γ are constants. For which values of γ do we get explosion?

Exercise 3. (Exit distribution of Bessel process) Let X be the solution of the SDE

$$dX_t = \frac{d-1}{2} \frac{1}{X_t} dt + dB_t \quad X_0 = x_0 > 0$$

where B is a standard Brownian motion and $d > 1$ is a constant.

- a) Find a non-constant function u such that $u(X_t)$ is a local martingale.
- b) Compute the ruin probability $\mathbb{P}(T_a < T_b)$ for $0 < a < b$ with $x_0 \in [a, b]$ where $T_a = \inf\{t \geq 0: X_t \leq a\}$ and $T_b = \inf\{t \geq 0: X_t \geq b\}$.
- c) Proceed similarly to determine the mean exit time $\mathbb{E}[T]$ where $T = \min(T_a, T_b)$.