

Stochastic Analysis – Problem Sheet 2.

Tutorial classes: Mon May 15th 16–18 in SemR 1.007. Claudio Bellani <claudio.bellani01@gmail.com>.
 Solutions will be collected Tuesday May 9th during the lecture. At most in groups of 3.

Exercise 1. (Constant quadratic variation) Let M be a continuous local martingale and $S \leq T$ two stopping times. Prove that $[M]_T = [M]_S < \infty$ a.s implies $M_t = M_S$ for all $t \in [S, T]$ a.s. . [Hint: consider the continuous local martingale $N_t = \int_0^t \mathbb{1}_{[S, T]}(s) dM_s$].

Exercise 2. (Feynman–Kac formula for Ito diffusions)

a) Consider the solution X of the SDE in \mathbb{R}^n

$$dX_t = b(t, X_t)dt + \sigma(t, X_t)dB_t, \quad X_0 = x,$$

where B is a d -dimensional Brownian motion and $b: \mathbb{R}^n \rightarrow \mathbb{R}^n$, $\sigma: \mathbb{R}^n \rightarrow \mathbb{R}^{n \times d}$ locally bounded continuous coefficients. Let \mathcal{L} be the associated infinitesimal generator. Fix $t > 0$ and assume that $\varphi: \mathbb{R}^n \rightarrow \mathbb{R}$ and $V: [0, t] \times \mathbb{R}^n \rightarrow \mathbb{R}_{\geq 0}$ are continuous functions. Show that any bounded $C^{1,2}$ solution $u: [0, t] \times \mathbb{R}^n \rightarrow \mathbb{R}$ of the equation

$$\begin{aligned} \frac{\partial}{\partial s} u(s, x) &= \mathcal{L}u(s, x) - V(s, x)u(s, x), & (s, x) \in (0, t] \times \mathbb{R}^n, \\ u(0, x) &= \varphi(x), \end{aligned}$$

has the stochastic representation

$$u(t, x) = \mathbb{E} \left[\varphi(X_t) \exp \left(- \int_0^t V(t-s, X_s) ds \right) \right].$$

In particular, there is at most only one solution of the PDE.

[Hint: show that $M_r = \exp(-\int_0^r V(t-s, X_s) ds) u(t-r, X_r)$ is a local martingale].

b) The price of a security is modeled by a geometric Brownian motion X with parameters $\alpha, \sigma > 0$:

$$dX_t = \alpha X_t dt + \sigma X_t dB_t, \quad X_0 = x > 0.$$

At price y we have a running cost of $V(y)$ per unit time. The total cost up to time t is then

$$A_t = \int_0^t V(X_s) ds.$$

Suppose that u is a bounded solution to the PDE

$$\begin{aligned} \frac{\partial}{\partial s} u(s, x) &= \mathcal{L}u(s, x) - \beta V(s, x)u(s, x), & (s, x) \in (0, t] \times \mathbb{R}_{\geq 0}, \\ u(0, x) &= 1, \end{aligned}$$

where \mathcal{L} is the generator of X . Show that the Laplace transform of A_t is given by

$$\mathbb{E}[e^{-\beta A_t}] = u(t, x).$$

Exercise 3. (Continuous Branching Process) Consider a family of diffusions $(X_t(x))_{t>0, x>0}$ satisfying the SDE

$$dX_t(x) = \alpha X_t(x)dt + \sqrt{\beta X_t(x)}dB_t, \quad X_0(x) = x,$$

where $\alpha \in \mathbb{R}$, $\beta \in \mathbb{R}_{>0}$. Existence of strong solutions to this equation follows from the Yamada–Watanabe theorem. Let (\tilde{X}, \tilde{B}) be an independent copy of (X, B) and let $Y_t(x, y) = X_t(x) + \tilde{X}_t(y)$ for $t > 0, x > 0, y > 0$.

a) (*Branching*) Compute the SDE satisfied by Y and prove that $(Y(x, y))_{t \geq 0}$ has the same law of $(X_t(x + y))_{t \geq 0}$. [Hint: use martingale characterization of weak solutions and pathwise uniqueness]

b) (*Duality*) Show that this implies that there exists a function $u: \mathbb{R}_{\geq 0} \times \mathbb{R}_{>0} \rightarrow \mathbb{R}_{\geq 0}$ such that

$$\mathbb{E}[e^{-\lambda X_t(x)}] = e^{-xu(t, \lambda)}, \quad x \in \mathbb{R}_{>0} \tag{1}$$

if we assume that the map $x \mapsto \mathbb{E}[e^{-\lambda X_t(x)}]$ is continuous.

c) Assume that $u: \mathbb{R}_{>0} \times \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}_{\geq 0}$ is differentiable with respect to its first parameter. Apply Ito formula to $s \mapsto G_s = e^{-u(t-s, \lambda)X_s(x)}$ and determine which differential equation u should satisfy in order for G to be a local martingale. Prove that in this case eq.(1) is satisfied (in particular, if a solution of the equation exists then it is unique).

d) (*Extinction probability*) Find the explicit solution u for the differential equation and using eq. (1) prove that if $\alpha = 0$ then

$$\mathbb{P}(X_t(x) = 0) = e^{-2x/(\beta t)}, \quad x, t > 0.$$