Institute for Applied Mathematics – **SS2017**
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Stochastic Analysis – Problem Sheet 4.

Tutorial classes: Mon May 29nd 16–18 in SemR 1.007. Claudio Bellani <`claudio.bellani01@gmail.com`>.
Solutions will be collected Tuesday May 23rd during the lecture. At most in groups of 3.

Exercise 1. (Passage time to a sloping line) Let X be a one–dimensional Brownian motion with $X_0 = 0$ and let $a > 0$, $b \in \mathbb{R}$.

a) Let $T_L = \inf \{t \geqslant 0 \colon X_t = a + b\, t\}$ denote the first passage time to the line $y = a + b\, t$. Show that

$$\mathbb{P}(T_L \leqslant t) = \mathbb{E}[e^{-bX_t - b^2 t/2} \mathbb{I}_{T_a \leqslant t}], \tag{1}$$

where $T_a = \inf \{t \geqslant 0 \colon X_t = a\}$ is the first passage time to level a.

b) Recall that, by the reflection principle, the law of T_a is absolutely continuous with density

$$f_{T_a}(t) = a\, t^{-3/2} \varphi\big(a / \sqrt{t}\big) \mathbb{I}_{(0,\infty)}(t),$$

where φ is the standard normal density. Deduce that the law of T_L is absolutely continuous with density

$$f_{T_L}(t) = a\, t^{-3/2} \varphi\big((a + b\, t) / \sqrt{t}\big) \mathbb{I}_{(0,\infty)}(t).$$

[Hint: in (1) take the conditional expectation w.r.t. $\mathcal{F}_{T_a}$].

c) Show that, for $b > 0$,

$$\mathbb{E}\Big[e^{-bX_t} \max_{s \leqslant t} (X_s)\Big] \simeq \frac{e^{b^2 t/2}}{2b}, \quad \text{and} \quad \mathbb{E}\Big[e^{bX_t} \max_{s \leqslant t} (X_s)\Big] \simeq b\, t e^{b^2 t/2}, \qquad \text{as } t \to \infty.$$

Exercise 2. Prove directly that the h-transform gives a transformation of martingale problems from $\mathrm{MP}(x_0, b, a)$ to $\mathrm{MP}(x_0, \tilde{b}, a)$ where $\tilde{b} = b + a\, \nabla(\log h) \nabla$. (That is, reproduce the argument of the notes without relying on the Itô decomposition of the process)

Exercise 3. Let $(X, \mathbb{P})$ be a solution of the Martingale Problem $\mathrm{MP}(x_0, b, a)$. Generalise appropriately the Girsanov transform to construct a measure $\mathbb{Q}$ under which the process X solves a martingale problem with a different drift. For simplicity, assume that all the necessary integrability conditions are satisfied. (What takes the place of the Brownian motion?)

Exercise 4. (Brownian Bridge) Let X be a d–dimensional Brownian motion with $X_0 = 0$.

a) Show that, for any $y \in \mathbb{R}^d$, the process

$$X_t^y = X_t - t\, (X_1 - y) \qquad t \in [0, 1]$$

is independent of X_1.

b) Let μ_y denote the law of X^y on $C([0, 1]; \mathbb{R}^d)$. Show that $y \mapsto \mu_y$ is a regular version of the conditional distribution of X given $X_1 = y$.