Stochastic Analysis – Problem Sheet 5.

Tutorial classes: Mon June 6th 16–18 in SemR 1.007. Claudio Bellani <`claudio.bellani01@gmail.com`>.
Solutions will be collected Tuesday May 30th during the lecture. At most in groups of 3.

Exercise 1. Use Girsanov transform to prove that the weak solution of the SDE

$$\mathrm{d}X_t = b_t(X)\mathrm{d}t + \mathrm{d}B_t$$

where $b: \mathbb{R}_{\geqslant 0} \times C(\mathbb{R}_{\geqslant 0}; \mathbb{R}^d) \to \mathbb{R}^d$ is a bounded, previsible drift, is unique in law.

Exercise 2. Let M be a positive continuous supermartingale such that $\mathbb{E}[M_0] < \infty$. Let $M_\infty = \lim_{t\to\infty} M_t$ (assumed to exist $\mathbb{P}$–a.s.). Show that if $\mathbb{E}[M_\infty] = \mathbb{E}[M_0]$ then M is a martingale and $\mathbb{E}[M_\infty|\mathcal{F}_t] = M_t$. *[Hint: prove that $\mathbb{E}[M_\infty|\mathcal{F}_t] \leqslant M_t$ and that $\mathbb{E}[M_t] = \mathbb{E}[M_0]$ and conclude.]*

Exercise 3. Assume that $\Omega = C(R_{\geqslant 0}; \mathbb{R}^d)$, $\mathbb{P}$ is the d–dimensional Wiener measure and that X is the canonical process on Ω and that the filtration $\mathcal{F}_\bullet$ is generated by X. Consider a predictable $\mathbb{R}^d$-valued drift b given by a function $b: \mathbb{R}_{\geqslant 0} \times \Omega \to \mathbb{R}^d$. By tilting $\mathbb{P}$ via $Z = \mathcal{E}(\int_0^\cdot b(X)\mathrm{d}X)$ we obtain that, under the tilted measure $\mathbb{P}^b$ the process X is a solution of the SDE

$$\mathrm{d}X_t = b_t(X) + \mathrm{d}W_t, \qquad t \geqslant 0$$

where W is a $\mathbb{P}^b$–Brownian motion.

a) Prove that if

$$|b_t(x)| \leqslant C(1 + |x_t|), \qquad t \geqslant 0, x \in \Omega,$$

then Novikov's condition holds conditionally on $\mathcal{F}_s$ for intervals $[s,t]$ such that $|t-s|$ is small enough, i.e.

$$\mathbb{E}\left[\exp\left(\frac{1}{2}\int_s^t |b_u(X)|^2 \mathrm{d}u\right)|\mathcal{F}_s\right] < +\infty.$$

b) Deduce that Z is a martingale. [Hint: prove that $\mathbb{E}[Z_t|\mathcal{F}_s] = Z_s$ for small time intervals $[s,t]$ and the conclude].

c) Prove that

$$\mathbb{P}(\|X\|_{[0,t]} > r) \leqslant 2\, d e^{-r^2/2dt} \qquad t \geqslant 0, r \geqslant 0.$$

where $\|X\|_{[0,t]}$ denotes the supremum wrt. the Euclidean norm of $(X_s)_{s\in[0,t]}$.

[Hint: use Doob's inequality for the submartingale $e^{\lambda X_t^i}$ and optimize over $\lambda > 0$]

d) Prove the same result as in (a) under the more general assumption that b is a previsible drift such that

$$|b_t(x)| \leqslant C(1 + \|x\|_{\infty,[0,t]}), \qquad t \geqslant 0, x \in \Omega$$

where $C < +\infty$.

Exercise 4. Given smooth, bounded functions $A: \mathbb{R}^d \to \mathbb{R}^d$, $V: \mathbb{R}^d \to \mathbb{R}$. Consider the operator on $L^2(\mathbb{R}^d)$ given by

$$H(A) = -\frac{1}{2}|\nabla - i A(x)|^2 + V(x)$$

We will assume that this operator is self–adjoint (with suitable domain), bounded from below and with discrete spectrum. We will denote $E_0(A)$ its smaller eigenvalue which we will assume simple (i.e. of multiplicity one). Let ψ the complex valued solution to

$$\partial_t \psi(t,x) = -H(A)\, \psi(t,x), \qquad \psi(0,x) = \psi_0(x),$$

which we will assume to exist, to be once differentiable in t and twice in x and be bounded with bounded derivatives.

a) Find a suitable functions $B, C: \mathbb{R}^d \to \mathbb{C}$ with which we can give the following Feynman–Kac representation for ψ:

$$\psi(t,x) = \mathbb{E}_x \left\{ \psi_0(X_t) \exp\left[\int_0^t B(X_s) \mathrm{d}X_s + \int_0^t C(X_s) \mathrm{d}s \right] \right\}$$

where under $\mathbb{E}_x$ the process X is a d-dimensional Brownian motion starting at $x \in \mathbb{R}^d$.

b) Prove that the lowest eigenvector of H_A is strictly positive everywhere.

c) Use the above representation to prove the *diamagnetic inequality*

$$E_0(A) \geqslant E_0(0).$$

[Hint: take $\psi_0(x) = 1$ and argue that $\psi(t,x) \simeq c e^{-E_0 t} \varphi(x) + o_t(1)$ where $H\varphi = E_0(A)\varphi$ and conclude]