Institute for Applied Mathematics – SS2017
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Stochastic Analysis – Problem Sheet 6.

Tutorial classes: Mon June 26th 16–18 in SemR 1.007. Claudio Bellani <`claudio.bellani01@gmail.com`>.
Solutions will be collected **Thursday June 22th** during the lecture. At most in groups of 3.

These exercises complement the lecture on June 13th, which is based on the first four sections of the paper

Beskos, Alexandros, Omiros Papaspiliopoulos, and Gareth O. Roberts. "Retrospective exact simulation of diffusion sample paths with applications." *Bernoulli* (2006): 1077-1098.

which can be retrieved from the link https://projecteuclid.org/euclid.bj/1165269151

Exercise 1. (Spatial Poisson Process)

Definition. *For $A \subset \mathbb{R}^2$, define $N(A) := \#A$ to be the cardinality of A, we define the set of locally finite point configurations*

$$N_{\text{lf}} := \{A \subset \mathbb{R}^2 : N(B \cap A) < \infty \text{ for all } B \subset \mathbb{R}^2 \text{ bounded}\},$$

furthermore, set $\mathcal{B}$ the Borel $\sigma-$algebra and $\mathcal{B}_0$ the set of all bounded Borel sets. Also define

$$\mathcal{N}_{\text{lf}} := \sigma(\{A \in N_{\text{lf}} : N(A \cap B) = m\} : B \in \mathcal{B}_0, m \in \mathbb{N} \cup \{0\}).$$

Now let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, then we call Φ a Point process if

$$\Phi : \Omega \to (N_{\text{lf}}, \mathcal{N}_{\text{lf}})$$

is measurable. This map induces a distribution

$$\mathbb{P}_\Phi(A) = \mathbb{P}(\{\omega \in \Omega : \Phi(\omega) \in A\}), \quad \text{for } A \in \mathcal{N}_{\text{lf}}.$$

In fact, the measurability of Φ is equivalent to $N(B)$ being a random variable.

We call Φ a homogeneous spatial Poisson point process with unit intensity on a set A with finite measure, if it in addition satisfies

- *For any $B \in \mathcal{B}$ with finite measure, $N(B) \sim \text{Poi}(\mu(B))$-The Poisson distribution with parameter $\mu(B)$*
- *For any $n \in \mathbb{N}$ and $B \in \mathcal{B}$ with $0 < \mu(B) < \infty$,*

$$\{\Phi_B | N(B) = n\} \sim \text{Bin}\left(n, \frac{\mu(B)}{\mu(A)}\right),$$

where we set $\mu(B) := \mathcal{L}^2(A \cap B)$. The second property says that if Φ is restricted to B and conditioned to have fixed $N(B) = n$, it is distributed as a Binomial point process.

a) Use the definition of the point process Φ to conclude Theorem 1 from the paper.

b) Define the probability generating functional

$$G_\Phi(u) := \mathbb{E}\left(\exp\left(\int_A \ln(u(x)) \mathrm{dN}(x) \right) \right) = \mathbb{E}\left(\prod_{x \in \Phi} u(x) \right)$$

Show the following identity

$$G_\Phi(u) = \exp\left(-\int_A (1 - u(x)) \mathrm{dx} \right).$$

Exercise 2. Prove the identity (8) in the paper. More precisely, let $W = \{W_t: \quad 0 \leqslant s \leqslant t\}$ be a Brownian Motion starting at 0, $m_T := \inf_{t \in [0,T]} \{W_t\}$ denote its minimum and $\theta_T := \sup\{t \in [0, T]: W_t = m_T\}$, the last time where the minimum is attained. For $a \in \mathbb{R}$, show the following identity

$$\mathbb{P}(m_T \in \mathrm{db}, \theta_T \in \mathrm{dt} | W_T = a) \propto \frac{b(b-a)}{\sqrt{t^3 (T-t)^3}} \exp\left(-\frac{b^2}{2t} - \frac{(b-a)^2}{2(T-t)} \right) \mathrm{dbdt},$$

where $\propto$ means equality up to multiplicative constants.

Exercise 3. Prove Proposition 3. This is concerned with the efficiency of the exact algorithms that were discussed in class.

Assume that we apply the exact algorithm(EA) on the time interval $[0, T]$ for some starting point $x \in \mathbb{R}$. Let ε denote the probability of accepting a proposed path and D the number of Poisson process points needed to decide whether to accept the proposed path. Let $N(T)$ be the total number of Poisson process points needed until the first path is accepted. Prove the following:

a) Consider EA 1, i.e. the case with $0 \leqslant \phi \leqslant M$. Then

$$\varepsilon \geqslant \exp(-M \cdot T), \qquad \mathbb{E}(D) = M \cdot T.$$

b) For EA 2, the case where $\phi \geqslant 0$ and wlog $\limsup_{x \to \infty} \phi(x) < \infty$ (one could just as easily consider the case $\limsup_{x \to -\infty} \phi(x) < \infty$) and $M(\omega) = M(\hat{m}) = \sup_{u \geqslant \hat{m}} \phi(u)$ as in the lecture, where $\hat{m}$ is the minimum of the path $\omega \sim \mathbb{Z}$, show

$$\varepsilon \geqslant \exp(-\mathbb{E}(M(\hat{m})) \cdot T) \qquad \mathbb{E}(D) = \mathbb{E}(M(\hat{m})) \cdot T.$$

In both cases

$$\mathbb{E}(N(T)) \leqslant \mathbb{E}(D) / \varepsilon.$$

Remark: In the case of EA 2 one would really want to bound $\mathbb{E}(M(\hat{m}))$. This is done in Proposition 4, but it is a bit trickier to prove.

Hint: Note that $\varepsilon = \mathbb{E}\big(\exp\big(-\int_0^T \phi(B_t(w))\big)\mathrm{dt}\big)$ and for b) use the convexity of $\exp(-\cdot)$.

For $\mathbb{E}(N(T))$, assume $\{\omega_i\}$ is a sequence of proposed paths, D_i is the number of Poisson process points needed to on the acceptance or rejection of the ith proposed path and $I = \inf\{i \geqslant 1: \omega_i \text{ is accepted}\}$. Set $\mathbb{E}(D|A)$ and $\mathbb{E}(D|A^c)$ the number of Poisson process points conditionally on accepting and rejecting the path respectively. Then

$$\mathbb{E}(N(T)) = \mathbb{E}\left(\sum_{i=1}^{I} D_i \right),$$

now estimate this by conditioning on I and using the "tower property".