Stochastic Analysis – Problem Sheet 7.

Tutorial classes: Mon July 10th 16–18 in SemR 1.007. Claudio Bellani <claudio.bellani01@gmail.com>.
Solutions will be collected Tuesday July 4th during the lecture. At most in groups of 3.

Exercise 1. Let $(\Omega, \mathcal{F})$ and $(\tilde{\Omega}, \tilde{\mathcal{F}})$ two measure spaces and $T: \Omega \to \tilde{\Omega}$ a measurable transformation. Prove that, for any two measures μ, ν on Ω we have

$$H(\mu \circ T^{-1} | \nu \circ T^{-1}) \leqslant H(\mu | \nu)$$

and, assuming that $H(\mu|\nu) < +\infty$, equality occours only if the density $\mathrm{d}\mu / \mathrm{d}\nu$ is a function of T.

In the following let $\Omega = C([0,1]; \mathbb{R}^d)$, $(\Omega, \mathcal{F}, \mathcal{F}_\bullet, \mu)$ be the standard d-dimensional Wiener space and $X: \Omega \times [0,1] \to \mathbb{R}^d$ the canonical process.

Exercise 2. Let γ be the standard Gaussian measure on $\mathbb{R}^d$ and σ a measure on $\mathbb{R}^d$ such that $\rho = \mathrm{d}\sigma / \mathrm{d}\gamma; \mathbb{R}^d \to \mathbb{R}_{>0}$ is a C^1 function with $\rho \geqslant \varepsilon$ for some $\varepsilon > 0$.

a) Prove that

$$H(\sigma|\gamma) = H(\nu|\mu) = \frac{1}{2} \mathbb{E}_\nu \|u\|^2_{L^2([0,1];\mathbb{R}^d)},$$

where ν is the measure on Ω with density $\rho(X_1)$ w.r.t μ and the drift u is given by $u_t = \nabla \log(P_{1-t}\rho)(X_t)$, where P is the transition operator for Brownian motion.

b) Prove that $\nabla \log(P_{1-t}\rho)(X_t)$ can be written as $\mathbb{E}_\nu[\nabla \log \rho(X_1) | \mathcal{F}_t]$ for $t \in [0,1]$ and deduce the log-Sobolev inequality for the Gaussian measure in $\mathbb{R}^d$:

$$H(\sigma|\gamma) \leqslant \frac{1}{2} \int_{\mathbb{R}^d} \left| \frac{\nabla \rho}{\rho} \right|^2 \mathrm{d}\sigma.$$

Exercise 3. Prove that if a family $(Y^\varepsilon)_\varepsilon$ satisfies the Laplace principle with rate function I then it satisfies also the Large Deviation principle with the same rate function, that is: for any open set $A \in \Omega$ and closed set $B \in \Omega$ we have

$$\liminf_{\varepsilon \to 0} \varepsilon \log \mathbb{P}(Y^\varepsilon \in A) \geqslant -\inf_{x \in A} I(x), \qquad \limsup_{\varepsilon \to 0} \varepsilon \log \mathbb{P}(Y^\varepsilon \in B) \leqslant -\inf_{x \in B} I(x).$$

a) For the lower bound approximate from below $\mathbb{I}_A$ with $\mathbb{I}_{B(x,\delta)}$ for some $x \in A$ where $B(x,\delta)$ is the ball with center x and radius δ. Then use that

$$\mathbb{E}[\exp(-\lambda(d(Y^\varepsilon, x) \wedge \delta) / \varepsilon)] \leqslant e^{-\lambda/\varepsilon} \mathbb{P}(d(Y^\varepsilon, x) > \delta) + \mathbb{P}(d(Y^\varepsilon, x) \leqslant \delta)$$

and estimate the r.h.s. with the Laplace principle and let $\lambda \to \infty$, $\delta \to 0$ at the right moment and use the lsc of I.

b) For the upper bound approximate $\mathbb{P}(Y^\varepsilon \in B) \leqslant \mathbb{E}[\exp(\lambda d(Y^\varepsilon, B) / \varepsilon)]$ and let $\lambda \to \infty$ at the right moment.

Exercise 4. Let Y^{ε} be the family of strong solutions to the SDE

$$\mathrm{d}Y^{\varepsilon} = b(Y_t^{\varepsilon})\mathrm{d}t + \sqrt{\varepsilon}\mathrm{d}X_t, \qquad t \in [0,1]$$

where b is bounded and Lipshitz. Prove that $(Y^{\varepsilon})_{\varepsilon}$ satisfies the Laplace principle with rate function I: $\Omega \to \mathbb{R}_{\geqslant 0}$ given by

$$I(x) = \frac{1}{2}\int_0^1 |\dot{x}(s) - b(x(s))|^2 \mathrm{d}s$$

if $\dot{x} \in L^2([0,1];\mathbb{R}^d)$ and $+\infty$ otherwise.

Exercise 5. (Bonus) Prove that the family of SDEs

$$\mathrm{d}Y^{\varepsilon} = b(Y_t^{\varepsilon})\mathrm{d}t + \sqrt{\varepsilon}\sigma(Y_t^{\varepsilon})\mathrm{d}X_t, \qquad t \in [0,1]$$

where b, σ are bounded and Lipshitz and $\sigma\,\sigma^T \geqslant \lambda I$ with $\lambda > 0$, satisfies the Laplace principle and identify the rate function.