# Stochastic Analysis – Problem Sheet 9.

Tutorial classes: Mon July 17th 16–18 in SemR 1.007. Claudio Bellani <claudio.bellani01@gmail.com>.
Solutions will be collected Tuesday July 11th during the lecture. At most in groups of 3.

**Exercise 1.** Consider the one dimensional SDE

$$\mathrm{d}X_t = -X_t^3 \mathrm{d}t + \mathrm{d}B_t, \qquad X_0 = x$$

where $B$ is a standard Brownian motion.

a) Let $f(t,x) = (1 + |x|^2)$ and $T_L = \inf\{t \geqslant 0 : |X_t| > L\}$. Use Ito formula to show that there exists a constant $\lambda$ such that the process $Z_t := e^{-\lambda(t \wedge T_L)} f(X_{t \wedge T_L})$ is a supermartingale.

b) Deduce that $\mathbb{P}(T_L \leqslant t) \to 0$ as $L \to \infty$.

c) Conclude that solutions of the SDE cannot explode (that is $\zeta := \sup_L T_L = \infty$ a.s.).

**Exercise 2.** Let $X, Y, Z$ be continuous semimartingale, prove the following iterative property for Stratonovich integrals. Let $I_t := \int_0^t Y_s \circ \mathrm{d}Z_s$ then

$$\int_0^t X_s \circ \mathrm{d}I_s = \int_0^t X_s Y_s \circ \mathrm{d}Z_s.$$

**Exercise 3.**

a) Solve the following Itô SDEs explicitly:

$$\mathrm{d}X_t = \frac{1}{2} X_t \,\mathrm{d}t + \sqrt{1 + X_t^2} \mathrm{d}B_t, \qquad X_0 = 0.$$

$$\mathrm{d}X_t = X_t(1 + X_t^2) \,\mathrm{d}t + (1 + X_t^2) \mathrm{d}B_t, \qquad X_0 = 1.$$

Do the solutions explode in finite time?

b) Solve explicitly

$$\mathrm{d}X_t = X_t^\gamma \mathrm{d}t + \alpha X_t \mathrm{d}B_t, \qquad X_0 = x > 0.$$

using the Doss-Sussmann method and determine the values of $\gamma$ for which explosion occours.

**Exercise 4.** Let $(X_t)$ be a $d$-dimensional stochastic process solving the SDE

$$\mathrm{d}X_t = b(X_t)\mathrm{d}t + \sum_{k=1}^{m} \sigma_k(X_k)\mathrm{d}B_t^k$$

where $B$ is an $m$-dimensional Brownian motion and $b, \sigma$ are bounded continuous vectorfields. Prove that, as $\eta\downarrow 0$,

a) $X_{t+h}$ converges to $X_t$ with strong $L^p$ order $1/2$;

b) $X_{t+h}$ converges to $X_t$ with weak order 1.

**Exercise 5.** If $c(t) = (x(t), y(t))$ is a smooth curve in $\mathbb{R}^2$ with $\mathrm{c}(0) = 0$,

$$A_t = \int_0^t (x(s)y'(s) - y(s)x'(s))\mathrm{d}s$$

describes the area that is covered by the secant from the origin to $c(s)$ in the interval $[0, t]$. Analogously, for a two-dimensional Brownian motion $B_t = (X_t, Y_t)$ with $B_0 = 0$, one defines the Lévy Area

$$A_t = \int_0^t (X_s \mathrm{d}Y_s - Y_s \mathrm{d}X_s).$$

a) Let $\alpha(t)$, $\beta(t)$ be $C^1$-functions, $p \in \mathbb{R}$, and

$$V_t = i\,p\,A_t - \frac{\alpha(t)}{2}(X_t^2 + Y_t^2) + \beta(t).$$

Use Itô formula to show that $e^{V_t}$ is a local martingale provided $\alpha'(t) = \alpha(t)^2 - p^2$ and $\beta'(t) = \alpha(t)$

b) Let $t_0 \geqslant 0$. Solutions to the equations for $\alpha, \beta$ with $\alpha(t_0) = \beta(t_0) = 0$ are

$$\alpha(t) = p\tanh(p(t_0 - t)), \qquad \beta(t) = -\log\cosh(p(t_0 - t)).$$

Conclude that

$$\mathbb{E}[e^{ipA_{t_0}}] = (\cosh(p t_0))^{-1}.$$

c) Show that the distribution of $A_t$ is absolutely continuous wrt Lebesgue with density

$$f_{A_t}(x) = (2\,t\cosh(\pi x / 2t))^{-1}, \qquad x \in \mathbb{R}.$$